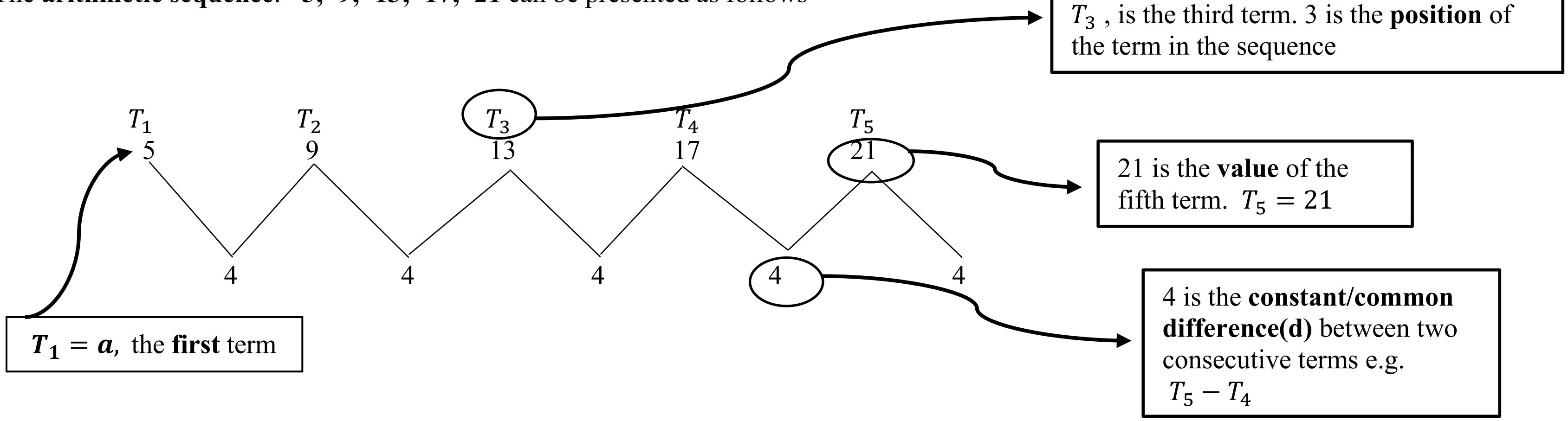
SUBJECT and GRADE	Mathematics Grade 12
TERM 1	Week 1
TOPIC	Sequences and Series
AIMS OF LESSON	• Recognise an Arithmetic Sequences • Find the general arithmetic sequence • Answer question based on the arithmetic sequence like finding the position in a sequence. • Find the sum of an arithmetic sequence • Sigma notation
RESOURCES	***Paper based resources***: Textbook chapter about Sequences and Series ***Digital resources***: https://www.youtube.com/watch?v=WE3S6OAwc-s
INTRODUCTION	In the previous grades you were introduced to numeric number patterns which is a sequence of numbers that follow a specific pattern. An example of a **linear pattern (arithmetic sequence)** is one where there is a **constant difference between consecutive terms.** In other words, the **same number** will be added to, or subtracted from each consecutive term.

A sequence is an ordered list of numbers or objects. A linear number pattern is also called an **ARITHMETIC SEQUENCE**

The **arithmetic sequence**: **5; 9; 13; 17; 21** can be presented as follows



An **Arithmetic sequence** is a sequence where the **difference between consecutive numbers or terms remain constant.**

In the sequence: **5; 9; 13; 17; 21; …**

$\boldsymbol{a}$ **= the first term** $= \boldsymbol{T_1}$

$\boldsymbol{d}$ **= constant difference**

$\boldsymbol{d} = T_2 - T_1 = T_3 - T_2$

$n = number\ of\ terms$

We see that $\boldsymbol{a = 5}$ and $\boldsymbol{d = 4}$

$\boldsymbol{T_1} = 5 = \quad \boldsymbol{a}$

$\boldsymbol{T_2} = 9 = 5 + 4 = \quad \boldsymbol{a + d}$

$\boldsymbol{T_3} = 13 = 5 + 2(4) = \boldsymbol{a + 2d}$

$\boldsymbol{T_4} = 17 = 5 + 3(4) = \boldsymbol{a + 3d}$

$\boldsymbol{T_n} = 5 + (n-1)(4) = \boldsymbol{a + (n-1)d}$

The **General Term** $\boldsymbol{T_n}$

Is given by

$$\boldsymbol{T_n = a + (n-1)d}$$

Example 1:	Solution:
Given the sequence: 2; 5; 8; …	**Constant difference:** $\boldsymbol{d = T_2 - T_1 = 5 - 2 = 3}$ $\boldsymbol{a = T_1 = 2}$
a) Determine the general term of the sequence.	$T_n = a + (n-1)d$ $= 2 + (n-1)3$ $= 2 + 3n - 3$ $= 3n - 1$ $\quad \therefore T_n = 3n - 1$
b) Use the general rule to determine the 40th term.	$T_n = 3n - 1$ $\therefore T_{40} = 3(40) - 1$ ← The **position** of the term is 40. Therefore, $n = 40$ $T_{40} = 119$
c) Which term in the pattern will be equal to 2012.	$T_n = 2012$ $\therefore T_n = 3n - 1 = 2012$ ← The **value** of the term is 2012. Therefore, $T_n = 2012$ $3n = 2012 + 1$ $3n = 2013$ $n = 671$ $\quad \therefore$ term number 671 of the sequence is equal to 2012

<u>Example 2:</u> Find the number of terms in the arithmetic sequence -2; -6; -10; … ; -150	<u>Solution:</u> $\boldsymbol{d} = -6-(-2) = \boldsymbol{-4}$ $T_n = a+(n-1)d$ $T_n = -2+(n-1)(-4)$ $T_n = -4n+2$ $T_n = -4n+2 = \boldsymbol{-150}$ $\therefore -4n = -152$ $\therefore n = 38$ There are 38 terms	<u>Example 3:</u> Determine the first three terms of an arithmetic sequence if the constant difference is 10 and the fourth term is 39.	<u>Solution:</u> Constant difference**:** $\boldsymbol{d = 10}$ $\boldsymbol{T_4 = 39}$ $T_n = a+(n-1)d$ $T_4 = a+(4-1)d$ $39 = a+3d$ $39 = a+3(10)$ $9 = a$ Hence the sequence is 9; 19; 29
<u>Example 4:</u> In an arithmetic sequence the 2^{nd} term is 9 and the 5^{th} term is 21. Determine a) The first three terms of the sequence. b) The 60^{th} term	$T_2 = a+d = 9$ (1) 2^{nd} term $T_5 = a+4d = 21$ (2) 5^{th} term $3d = 12$ **(2) − (1)** $\boldsymbol{d = 4}$ $\therefore T_2 = a+d = 9$ $a+4 = 9$ $\boldsymbol{a = 5}$ First three terms are 5; 9; 13; $T_{60} = a+59d = 5+59(4) = 241$ Hence the 60^{th} term is 241	<u>Example 5:</u> $2p-3;\ 3p-1;\ 5p-2$ are the first three terms of an arithmetic sequence. a) Determine the value of p. b) The first three terms of the sequence. c) Determine the term equal to 2013	<u>Solution:</u> a) $\boldsymbol{d} = T_2 - T_1 = T_3 - T_2$ $(3p-1)-(2p-3) = (5p-2)-(3p-1)$ $3p-1-2p+3 = 5p-2-3p+1$ $p+2 = 2p-1$ $\boldsymbol{p = 3}$ b) Replacing $p = 3$ in the sequence we have the first three terms as 3; 8; 13 c) $T_n = a+(n-1)d = 2013$ $3+(n-1)(5) = 2013$ $3+5n-5 = 2013$ $5n = 2015$ $n = 403$
CAN YOU?	1) Given the following sequence: 3; 8; 13; 18; … Determine: a) The general term. b) The 20^{th} term. c) Which term of the sequence is equal to 223? 2) In an arithmetic sequence, $T_3 = -2$ and $T_8 = 23$. Determine the first term and the constant difference. 3) Find the number of terms in the arithmetic sequence $-5;\ -11;\ -17;\ \ldots;\ -491$ 4) The first three terms of an arithmetic sequence are, $x-8;\ x;\ 2x-5$. Determine a) The value of x. b) The general term. c) The value of the 115^{th} term.		<u>Solutions:</u> 1) a) $T_n = 5n-2$ b) 98 c) $n = 45$ 2) $d = 5$ $a = -12$ 3) 82 4) a) 13 b) $T_n = 8n-3$ c) 917

SERIES: A series is created by **adding the terms of a sequence.**

2; 5; 8; 11; 14;	**Arithmetic Sequence**
2 + 5 + 8 + 11 + 14	**Arithmetic Series**

The **Sum of a sequence** is labelled as $\boldsymbol{S_n}$ or the Greek symbol

$$\sum$$

In the series 2 + 5 + 8 + 11 + 14 + …

$$\boldsymbol{S_1} = T_1 = 2$$
$$\boldsymbol{S_2} = T_1 + T_2 = \quad \boldsymbol{S_1} + \boldsymbol{T_2} = 2 + 5 = 7$$
$$\boldsymbol{S_3} = T_1 + T_2 + T_3 = \quad \boldsymbol{S_2} + \boldsymbol{T_3} = 2 + 5 + 8 = 15$$
$$\boldsymbol{S_4} = T_1 + T_2 + T_3 + T_4 = \boldsymbol{S_3} + \boldsymbol{T_4} = 2 + 5 + 8 + 11 = 26$$

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$$\boldsymbol{S_n} = \boldsymbol{S_{n-1}} + \boldsymbol{T_n}$$

$$\boxed{\boldsymbol{S_n} = \boldsymbol{S_{n-1}} + \boldsymbol{T_n}}$$

Let $\boldsymbol{T_n = a + (n-1)d = l}$ the last term

Then
$$S_n = a + (a+d) + (a+2d) + \ldots + (l-d) + l$$
$$S_n = l + (l-d) + (l-2d) + \ldots + (a+d) + a$$
$$2S_n = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l)$$
$$2S_n = n(a+l)$$
$$\boldsymbol{S_n = \frac{a}{2}(a+l)}$$
$$S_n = \frac{a}{2}(a + \boldsymbol{a} + (\boldsymbol{n}-\boldsymbol{1})\boldsymbol{d}) = \boldsymbol{\frac{n}{2}[2a + (n-1)d]}$$

Hence the **sum of the first $\boldsymbol{n}$ terms** of an **arithmetic series** is given by the formula:

$$\boxed{\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d]} \quad \textbf{or} \quad \boldsymbol{\frac{n}{2}[a + l]}\;;\ \textbf{where } \boldsymbol{l}\textbf{, is the last term}}$$

Example 6:

Consider the arithmetic series $(-1)+\left(\frac{-3}{2}\right)+(-2)+\cdots+(-16)$.

a) Determine the number of terms in this series.

Solution:

$T_n = a+(n-1)d$

$T_n = -1+(n-1)\left(-\frac{1}{2}\right) = -16$

$\therefore\ -1-\frac{1}{2}n+\frac{1}{2} = -16$

$\therefore\ -\frac{1}{2}n = -16+\frac{1}{2}$

$\therefore\ n = 31$

b) Calculate the sum of the series.

Solution:

$S_n = \frac{n}{2}[a+l]$

$\therefore S_{31} = \frac{31}{2}[-1+(-16)]$

$\therefore\ S_{31} = -\frac{527}{2}$

Example 7:

How **many terms** of the arithmetic series $1+4+7+\cdots$ will **add up to 145?**

Solution:

$a=1;\ d=3;\ n=?;\ S_n=145$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$145 = \frac{n}{2}[2(1)+(n-1)3]$

$290 = n(2+3n-3)$

$290 = n(3n-1)$

$0 = 3n^2-n-290$

$0 = (3n+29)(n-10)$

$n = -\frac{29}{3}$ or $n = 10$

$\therefore n = 10\ ;\ n \in N$

Example 8:

Consider the arithmetic series $-4-1+2+\cdots$

Calculate the smallest value of n for which $S_n > 300$

Solution:

$a=-4;\ d=3;\ n=?;$ let $S_n = 300$

$S_n = \frac{n}{2}[2a+(n-1)d] = 300$

$\frac{n}{2}[2(-4)+(n-1)3] = 300$

$\therefore\ n[3n-11] = 600$

$\therefore\ 3n^2-11n-600 = 0$

$\therefore\ n = \frac{-(11)\pm\sqrt{(-11)^2-4(3)(-600)}}{2(3)}$

$\therefore\ n = 16{,}09$ or $n = -12{,}43$

$\therefore$ The smallest possible value of n is 17.

SIGMA NOTATION: The Greek letter $\sum$ **Sigma means the sum of.**

It is used to denote the sum of a set of consecutive terms of a sequence or series. In this notation we have to indicate the position of the first and last term of the sequence/series which are added.

$$\sum_{k=1}^{n} T_k = T_1 + T_2 + T_3 + \cdots + T_n = S_n$$

This is read as, "the sigma or sum of T_k, from $k = 1$ to $k = n$.

This means, take the sum of the terms of a sequences/series from the first term to the n'th term of the sequence/series.

Or it is read as the sum of the first n terms of the sequence/series.

n is the number or position of the last term of the set of numbers which is being added. Lets refer to it as ***last/top value***

$$\sum_{k=m}^{n} T_k$$

General term expressed in terms of k

m is the number or position of the first term of the set of numbers which is being added. Lets refer to it as ***bottom value***

n, the number of terms which are added:

$$n = top\ value - bottom\ value + 1$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \sum_{k=1}^{n}[a + (k-1)d]$$

<u>Example 9:</u> Determine the value of:

$$\sum_{n=1}^{5}(3n+2)$$

Substitute, $n = 1$, in general term up to, $n = 5$.

<u>Solution:</u>
Method 1

$$S_n = \sum_{n=1}^{5}(3n+2)$$

$S_5 = (3.1 + 2) + (3.2 + 2) + (3.3 + 2) + (3.4 + 2) + (3.5 + 2)$
$= 5 + 8 + 11 + 14 + 17$
$S_5 = 55$

<u>Example 10</u>: Determine the value of:

$$\sum_{k=4}^{7} 2k$$

<u>Solution:</u>
Method 1

Number of terms: Top – bottom+1 $= (7 - 4 + 1) = 4$

$$\sum_{k=4}^{7} 2k$$

$S_4 = 2(4) + 2(5) + 2(6) + 2(7)$
$= 8 + 10 + 12 + 14$
$S_4 = 44$

Method 2:	**Method 2:**
$$\sum_{n=1}^{5}(3n+2)$$ Substitute, $n = 1, 2, 3$. To establish the pattern, is it an Arithmetic or not $(3.1+2)+(3.2+2)+(3.3+2)+\cdots.$ $= 5+8+11+\cdots\ldots..$ $8-5=3$ and $11-8=3$ ∴ it is Arithmetic. $\therefore \quad S_n = \frac{n}{2}[2a+(n-1)d]$ $\therefore \quad S_5 = \frac{5}{2}[2(5)+4(3)]$ $\therefore \quad S_5 = 55$ Number of terms is 5 or $Top - bottom + 1 = (5-1+1)$	$$\sum_{k=4}^{7}2k$$ Substitute, $n = 1, 2, 3$. To establish the pattern, is it an Arithmetic or not $2(4)+2(5)+2(6)+\cdots$ $= 8+10+12+\cdots\ldots..$ $10-8=2$ and $12-10=2$ ∴ it is Arithmetic. $\therefore \quad S_4 = \frac{4}{2}[2a+3d]$ $\therefore \quad S_4 = 2[2(8)+3(2)]$ $\therefore \quad S_4 = 44$ Number of terms: Top – bottom+1 $= (7-4+1) = 4$

Note the two methods done above. If the sum of a set of terms of a sequence/series is for many terms say a hundred terms, it will make more sense to use the second method as you will be able to determine the sum without needing to calculate the value of all the terms and then to add them.

Example 11: Write the following series in sigma notation: $5+8+11+14+17$

Solution:

1) First calculate the general term for the series where $\boldsymbol{a = 5}$ **and** $\boldsymbol{d = 3}$.

Hence $T_n = a+(n-1)d$
$T_n = 5+(n-1)3$
$T_n = 5+3n-3$
$T_n = 3n+2$

2) Write the formula now in sigma notation

$$\sum_{n=\cdots}^{\cdots}(3n+2)$$

Bottom value is the first term which is equal to 5:
$3n+2 = \mathbf{5}$
$3n = 3$
$n = 1$

Top value is the last term which is equal to 17:
$3n+2 = \mathbf{17}$
$3n = 15$
$n = 5$

3) $$\sum_{n=1}^{5}(3n+2)$$

<table>
<tr><td>CAN YOU?</td><td>1) Determine $5 + 12 + 19 + \cdots + 54$

2) How many terms of the arithmetic series $3 + 7 + 11 + \cdots$ will add up to 210?

3) Determine the value of the following
$$\sum_{r=0}^{10}(2r+5)$$
4) Write the following in sigma notation $7 + 10 + 13 + \cdots + 25$</td><td>Answers:
1) 236

2) 10

3) 165

4)
$$\sum_{n=0}^{6}(7+3n)$$</td></tr>
<tr><td>ACTIVITIES/ASSESSMENT</td><td colspan="2"><table>
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<tr><td>CONSOLIDATION</td><td colspan="2">Arithmetic Sequence
$T_1,\ T_2\ ,\ T_3,\ T_4, \ldots\ T_n$
$T_n = a + (n-1)d$ where $\boldsymbol{a} = T_1$ and $\boldsymbol{d} = T_2 - T_1 = T_3 - T_2$

Arithmetic Series
$T_1 + T_2 + T_3 + T_4 + \cdots + T_n = S_n$
$S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}[a + l]$

Sigma Notation
$$\sum_{k=1}^{n} \boldsymbol{T_k} = \boldsymbol{S_n}$$</td></tr>
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