SUBJECT and GRADE	Mathematics Grade 12	
TERM 1	Week 2	
TOPIC	Geometric Sequences and Series	
AIMS OF LESSON	• Recognise a geometric sequence • Find the general term of a geometric sequence • Use the general term formula to determine the position of a term and finding a term in a given position • The sum of a geometric series • Sigma Notation	
RESOURCES	***Paper based resources***	***Digital resources***
	Your textbook and Mind the Gap, Page 42	https://www.youtube.com/watch?v=BFgNEy8rAMU

INTRODUCTION

Let's explore the following number patterns.

1. 2 ; 8; 32; 128; …

$\times 4 \quad \times 4 \quad \times 4$

Please note: a is the first term and r is the common ratio

Therefore:

$a = 2$ and $r = 4$

You will agree that each term is **multiplied** by a certain constant. This value is known as the common ratio. (r)

This constant can be determined by:

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_{n+1}}{T_n}$$

Write down the value of a and r in each of the following cases.

2. 1; -2; 4; -8; …
3. 2; -1; $\frac{1}{2}$; $-\frac{1}{4}$; …
4. 243; 81; 27; …

Answer

2. $a = 1; r = -2$
3. $a = 2; r = -\frac{1}{2}$
4. $a = 243; r = \frac{1}{3}$

CONCEPTS AND SKILLS

Let explore this further:

1. 2; 8; 32; 128; …

$T_1 = 2 = a$
$T_2 = 2 \times 4 = ar$
$T_3 = 2 \times 4 \times 4 = ar^2$
$T_4 = 2 \times 4 \times 4 \times 4 = ar^3$
$T_5 = \qquad\qquad = ar^4$
$\therefore T_{10} = ar^9$

Take note of the pattern….

The number of the term and the power of r

This formula can be used to determine any term in this type of sequence.

$T_n = ar^{n-1}$ T_n is the general term

We are going to use this rule to determine GENERAL TERM of a geometric sequence.

Example 1. Determine the general term of the sequence
2; 8; 32; 128; ...

Solution:

$T_n = ar^{n-1}$
$T_n = 2(4^{n-1})$
$T_n = 2(2^2)^{n-1}$
$T_n = 2(2^{2n-2})$
$T_n = 2^{2n-1}$

Method

1. Find the value of a. **[First term]**
2. Determine the value of r. **[Common ratio]**
3. Substitute into the T_n formula
4. Simplify

Information needed

$a = 2;$

$r = \frac{T_2}{T_1} = \frac{8}{2} = 4$

$T_n = ?$

Some advice: Check your rule:
Eg. $T_4 = 2^{2(4)-1} = 2^7 = 128$
Please note that this is **NOT** equal

$T_n = 2(4^{n-1}) \neq 8^{n-1}$

Exponential laws must be applied to simplify it further.

CAN YOU?

Find the general term of each of the following sequences:

(a) 4; 12; 36; ….

(b) 1; -2; 4; -8; …

(c) 2; 6; 18; ….

(d) 128; 64; 32; …

(e) $\frac{2}{3}$; -2; 6; ...

(f) 0,25; 0,5; 1; …

Answer

(a) $T_n = 4(3)^{n-1}$

(b) $T_n = (-2)^{n-1}$

(c) $T_n = 2(3)^{n-1}$

(d) $T_n = 2^{8-n}$

(e) $T_n = \frac{2}{3}(-3)^{n-1}$

(f) $T_n = (2)^{n-3}$

Example 2: Find the 7th term of the sequence

$$24;\ 6;\frac{3}{2};\frac{3}{8};\ \ldots\ldots$$

Solution:

$$T_n = ar^{n-1}$$
$$T_7 = 24\left(\frac{1}{4}\right)^{7-1}$$
$$T_7 = 24\left(\frac{1}{4}\right)^{6}$$
$$\mathrm{T}_7 = \frac{3}{512}$$

Use your calculator for the final answer!

Rough work

$a = 24$

$r = \frac{6}{24} = \frac{1}{4}$

$n = 7$

$T_n = ?$

Substitute

CAN YOU?

2. In the following sequences determine the:

(a) 11^{th} term of $-3;\ 6;\ -12;\ \ldots.$

(b) 9^{th} term of $2; 8; 32; \ldots$

(c) 25^{th} term of $a; ab; ab^2; \ldots$

Answer

(a) $T_n = -3072$

(b) $T_n = 131\ 072$

(c) $T_n = ab^{24}$

Example 3:

Which term in the sequence:

$12; 4; \frac{4}{3}$ is equal to $\frac{4}{243}$?

Solution:

$$T_n = ar^{n-1}$$
$$\frac{4}{243} = 12\left(\frac{1}{3}\right)^{n-1}$$
$$\frac{1}{729} = \left(\frac{1}{3}\right)^{n-1}$$
$$\left(\frac{1}{3}\right)^{6} = \left(\frac{1}{3}\right)^{n-1}$$
$$\therefore n - 1 = 6$$
$$n = 7$$

Rough work on RHS of page

$a = 12;$

$r = \frac{4}{12} = \frac{1}{3}$

$n = ?$

$T_n = \frac{4}{243}$

I must find n.

Substitute and simplify

CAN YOU?

3. Which term of the geometric sequence $2; 6; 18; \ldots.$ is equal to 4374?

4. Find the number of terms in the sequence $2; 4; 8; \ldots.1024.$

5. How many terms are there in the sequence $5x; 20x; 80x; \ldots 20480x$?

Answer

3. $n = 8$

4. $n = 10$

5. $n = 7$

Example 4:

Determine the first 4 terms of a geometric sequence where the 2nd term is 10 and the 9th term is $\frac{5}{64}$.

T_2: $ar = 10$ Equation 1

T_9: $ar^8 = \frac{5}{64}$ Equation 2

Given info

$T_2 = 10$

$T_9 = \frac{5}{64}$

You will need a and r to write down the first 3 terms

$\frac{Equ\ 2}{Equ\ 1}$: $\frac{ar^8}{ar} = \frac{5}{64} \div 10$

$$r^7 = \frac{5}{64} \times \frac{1}{10}$$

$$r^7 = \frac{5}{640}$$

$$r = \sqrt[7]{\frac{5}{640}} = \frac{1}{2}$$

Substitute into Equation 1:

$$a(\frac{1}{2}) = 10$$

$$a = 10 \div \frac{1}{2}$$

$$a = 20$$

The three terms are:

$20; 10; 5; \frac{5}{2}$

See the confirmation of term 2.

CAN YOU?

6. Determine the geometric sequence whose 4th term is 24 and 7th term is 192.

Answer

6. $r = 2\ ;\ a = 3$

   $3; 6; 12; 24; ....$

Example 5. The first 3 terms of the geometric sequence are given as:

$$x-4;\ \ x;\ \ \ x+12$$

(a) Determine the value of x.
(b) Determine the value of the first 3 terms
(b) Determine the value of 10th term.

Solution:

(a)

$$\frac{T_2}{T_1}=\frac{T_3}{T_2} \qquad [r=]$$

$$\frac{x}{x-4}=\frac{x+12}{x}$$

$$x^2=(x-4)(x+12)$$

$$x^2=x^2+8x-48$$

$$8x-48=0$$

$$8x=48$$

$$x=6$$

We need an equation to solve for x

Thus we require, two things which are equal.

In this case the **ratio is equal**

(b) The terms are:

$$x-4;\ \ x;\ \ \ x+12$$
$$=6-4;6;6+12$$
$$=2;\ \ 6;18\ldots.$$

(c) $T_n=ar^{n-1}$

$$T_{10}=2(3)^9$$
$$=39366$$

CAN YOU?

7. $x-4;x+2;3x+1;\ldots$ are the first three terms of a geometric sequence. Determine the sequence if x is positive.

8. $t+1;1-t;1-5t;\ldots$ are the first three terms of a geometric sequence.
(a) Determine the numeric value of t where $t\neq 0$.
(b) Determine the sequence.
(c) Determine the 10th term.
(d) Which term equals $10\frac{2}{3}$?

Answer

7. $4;10;25;\ldots.$

8.
(a) $t=-\frac{1}{3}$
(b) $\frac{2}{3};\frac{4}{3};\frac{8}{3}$
© $T_{10}=\frac{1024}{3}$
(d) $n=5$

THE SUM OF A SERIES

This time we are going to determine the sum of a geometric series.

$a = 1$
$r = 2$
$n = 5$

Let's see if we can find an easy way to do the following series:

$1 + 2 + 4 + 8 + 16$

This is an easy example. The sum is 31. But what if there are 10 or more terms in a series. Let's take a closer view to observe a pattern.

$S_5 = 1 + 2 + 4 + 8 + 16$ (1)

$2S_5 = \quad 2 + 4 + 8 + 16 + 32$ (2)

Multiply Equation (1) by r.

(1) – (2): $S_5 - 2S_5 = 1 - 32$

$-S_5 = -31$

$S_5 = 31$

Now we will be doing it in general terms: ***[You need to know this for examination purposes]***

$\boldsymbol{S_n = a + ar + ar^2 + ar^3 + \cdots ar^{n-1}}$

$\boldsymbol{rS_n = \quad ar + ar^2 + ar^3 + \cdots \quad ar^n}$

$\boldsymbol{rS_n - S_n = ar^n - a}$

$\boldsymbol{S_n(r-1) = a(r^n - 1)}$

$\boldsymbol{S_n = \frac{a(r^n-1)}{r-1}}$

$S_n = \frac{a(r^n - 1)}{r - 1}$

Now we be using this formula to determine the sum of a geometric series.

Example 1:

Determine the sum of:

$1 + 2 + 4 + 8 \ldots$. To 6 terms

Solution:

$a = 1$
$r = 2$
$n = 6$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_6 = \frac{1(2^6 - 1)}{2 - 1}$

$S_6 = 63$

Example 2:

In the geometric series:

$-243 - 81 - 27; \ldots . - \frac{1}{81}$

(a) Determine the number of terms.

(b) What is sum of this series?

$a = -243$

$r = \frac{-81}{-243} = \frac{1}{3}$

$T_n = -\frac{1}{81}$

$n = ?$

Solution:

(a)

$$T_n = ar^{n-1}$$

$$-\frac{1}{81} = (-243)\left(\frac{1}{3}\right)^n$$

$$\left(\frac{1}{3}\right)^n = -\frac{1}{81} \div \frac{-243}{1}$$

$$\left(\frac{1}{3}\right)^n = -\frac{1}{81} \times -\frac{1}{243}$$

$$\left(\frac{1}{3}\right)^n = \frac{1}{19683}$$

$$\left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^9$$

$$n = 9$$

(b)

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_9 = \frac{-243((\frac{1}{3})^9 - 1)}{\frac{1}{3} - 1}$$

$$S_9 = -\frac{9841}{27}$$

Remember to use the  **effectively!**

You can use the FACT [factors] function on the calculator to determine the prime factors

CAN YOU?

1. Find:
(a) The number of terms in each series.
(b) The sum of each series.

(i) $4 + 12 + 36 + \cdots + 78732$

(ii) $2 - 8 + 32 - \cdots - 32768$

(iii) $81 + 27 + 9 + \cdots . + \frac{1}{27}$

Answer:

(i) $n = 10$; $S_n = 118\,096$

(ii) $n = 8$; $S_n = -26214$

(iii) $n = 8$; $S_n = \frac{3280}{27}$

Example 3:

Expand and then calculate:

$$\sum_{r=3}^{15} 3(-2)^{r-1}$$

Solution:

$$\sum_{r=3}^{15} 3(-2)^{r-1} = 3(-2)^2 + 3(-2)^3 + 3(-2)^4 + \cdots + 3(-2)^{14}$$

$$= 12 - 24 + 48 + \cdots + 49152$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{13} = \frac{12((-2)^{13} - 1)}{-2 - 1} = 32772$$

Substitute to expand

$a = 12$
$r = -2$

$n = 15 - 3 + 1$

$n = 13$
$S_n = ?$

2. Find the sum of the following:

$$(a) \quad \sum_{n=1}^{3} 128\left(\frac{1}{2}\right)^{n-1}$$

$$(b) \quad \sum_{k=0}^{4} (-3)^k$$

3. What is the value of m for which

$$\sum_{k=1}^{m} 5(3)^{k-1} = 65?$$

Answer:

2.(a) 224

(b) -20

3. $m = 3$

ACTIVITIES/ASSESSMENT

Textbook	Mind Action Series	Everything Maths Siyavula	Classroom Mathematics	Platinum
Geometric Sequences	Ex: 4 Pg:9	Ex: 1.4- 1.61 Pg: 14-18	Ex: 1.4 Pg: 11	Ex:3-4 Pg: 8-9
Sum Geometric Series	Ex: 5 Pg:23	Ex: 1.9 Page 35	Ex: 1.6 Pg: 19	Ex: 6 Pg:14

CONSOLIDATION

Summary