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| SUBJECT and GRADE | Mathematics Grade 12 | |
| TERM 1 | Week 3 | |
| TOPIC | Geometric Sequences and Series | |
| AIMS OF LESSON | • Sum to Infinity<br>• Application sequence and Series | |
| RESOURCES | ***Paper based resources*** | ***Digital resources*** |
| | Your textbook and Mind the Gap, Page 56 | https://www.youtube.com/watch?v=btve4T1R8Og<br>https://www.youtube.com/watch?v=MTOKAA8rRA0 |

**INTRODUCTION:** We have been looking at geometric series with a definite number of terms. Now we are going to look at the sum of a geometric series with an **infinite** $(\infty)$ number of terms

Remember the sum for Geometric Series**:**

$$\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}} \text{ or } \boldsymbol{S_n = \frac{a(r^n-1)}{r-1}}$$

Compare the sum of the following geometric series:

**Example 1**

$\mathbf{1+2+4+8+\cdots}$

$S_2 = 1+2 \quad = 3$

$S_3 = 1+2+4 \quad = 7$

$S_4 = 1+2+4+8 \quad = 15$

$S_5 = 1+2+4+8+16 \quad = 31$

$S_6 = 1+2+4+8+16+32 \quad = 63$

$S_9 = \frac{a(r^n-1)}{r-1} = \frac{1(1-2^9)}{1-2} \quad = 511$

**Example 2**

$\mathbf{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}$

$S_2 = 1\frac{1}{2} \quad = 1{,}5$

$S_3 = 1+\frac{1}{2}+\frac{1}{4} = 1\frac{3}{4} \quad = 1{,}75$

$S_4 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1\frac{7}{8} \quad = 1{,}875$

$S_5 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = 1\frac{15}{16} \quad = 1{,}9375$

$S_6 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} = 1\frac{31}{32} \quad = 1{,}984375$

**Example 1 continues**

$$S_{20} = \frac{1(2^{20}-1)}{2-1} \qquad = 1\ 048\ 575$$

**As $\boldsymbol{n}$ increases, $\boldsymbol{S_n}$ becomes very large**. Mathematically we say that:

**If $\boldsymbol{n \to \infty}$ then $\boldsymbol{S_n \to \infty}$.**

$\therefore$ This series **diverges.**

A **geometric series will diverge** (the sum will approach a very large positive or negative value) as $n$ approaches infinity.

**Example 3**

Determine:

$$\sum_{n=1}^{\infty} 2.\left(\frac{1}{2}\right)^{n-1}$$

**Example 2 continues**

$$S_9 = \frac{a\left(1-\left(\frac{1}{2}\right)^9\right)}{1-r} = \frac{1\left(1-\frac{1}{512}\right)}{\frac{1}{2}} \qquad = 1{,}996\ldots \quad \approx 2$$

**As $\boldsymbol{n}$ increases, $\boldsymbol{S_n}$ approaches 2**. Mathematically we say that:

**If $\boldsymbol{n \to \infty}$ then $\boldsymbol{S_n \to 2}$.**

$\therefore$ **This series converges to 2.**

A **geometric series will converge** (the sum will approach a specific value) as $n$ approaches infinity, if the **constant ratio is a number between $\boldsymbol{-1}$ and 1.**

**Thus**

**Convergent geometric series**

$$\boldsymbol{-1 < r < 1 \quad \text{and} \quad r \neq 0}$$

**If $\boldsymbol{-1 < r < 1}$**, then $\boldsymbol{r^n \to 0}$ when $n \to \infty$

$$\therefore\ S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a(1-0)}{1-r}$$

$$S_\infty = \frac{a}{1-r}$$

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

**Example 4**

For which values of $x$ will the following series converge?

$$(x+1)+(x+1)^2+(x+1)^3+\cdots$$

**Solution to Example 3:**

$$\sum_{n=1}^{\infty} 2.\left(\frac{1}{2}\right)^{n-1} = 2\left(\frac{1}{2}\right)^{0} + 2\left(\frac{1}{2}\right)^{1} + 2\left(\frac{1}{2}\right)^{2} + \cdots$$

$$= 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

$a = 2$ and $r = \frac{1}{2}$

$$S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{2}} = 2 \div \frac{1}{2} = 2 \times \frac{2}{1} = 4$$

**Solution to Example 4:**

Convergent series**:**

$$r = \frac{T_2}{T_1} = \frac{(x+1)^2}{(x+1)} = x + 1$$

For the series to converge**,** $-1 < r < 1$

$$\therefore\ -1 < x + 1 < 1$$
$$\therefore\ -2 < x < 0$$

**Example 5**

Given the geometric series: $256 + p + 64 - 32 + \cdots$

a) Determine the value of $p$.

b) Calculate the sum of the first 8 terms of the series.

c) Why does the sum to infinity for this series exist?

d) Calculate $S_\infty$.

**Solution:**

a) $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3}$

$$\frac{p}{256} = \frac{-32}{64}$$

$$p = -128$$

b) $S_n = \frac{a(1-r^n)}{1-r}$ , $r = \frac{T_4}{T_3} = -\frac{1}{2}$

$$S_8 = \frac{256\left(1 - \left(-\frac{1}{2}\right)^8\right)}{1 - \left(-\frac{1}{2}\right)} = 170$$

c) $-1 < r < 1$ **(see next page)**

**CAN YOU?**

1) Calculate the sum to infinity for
   (a) $16 + 8 + 4 + \ldots$
   (b) $48 - 12 + 3 - \cdots$

2) Determine:

$$\sum_{n=1}^{\infty} \mathbf{2.(4)^{1-n}}$$

3) Determine the value of $x$ for which the following series converges.
   a) $x + 2x^2 + 4x^3 + \cdots$
   b) $(2x-5) + (2x-5)^2 + (2x-5)^3 + \cdots$

4) Consider the infinite geometric series:
   $45 + 40{,}5 + 36{,}45 + \cdots$

   a) Calculate the value of the TWELTH term of the series (correct to two decimal places).
   b) Explain why the series converges.
   c) Calculate the sum to infinity of the series.

d) $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{256}{1-\left(-\frac{1}{2}\right)} = \frac{512}{3}$

**Answers:**

1) a) 32 b) $\frac{192}{5}$ **(see next page)**

2) $\frac{8}{3}$

3) a) $-\frac{1}{2} < x < \frac{1}{2}$ b) $2 < x < 3$

4) a) 14, 12 b) $-1 < r < 1$ ; $(r \neq 0)$ c) 450

**MIXED APPLICATIONS:**

**Example 6**

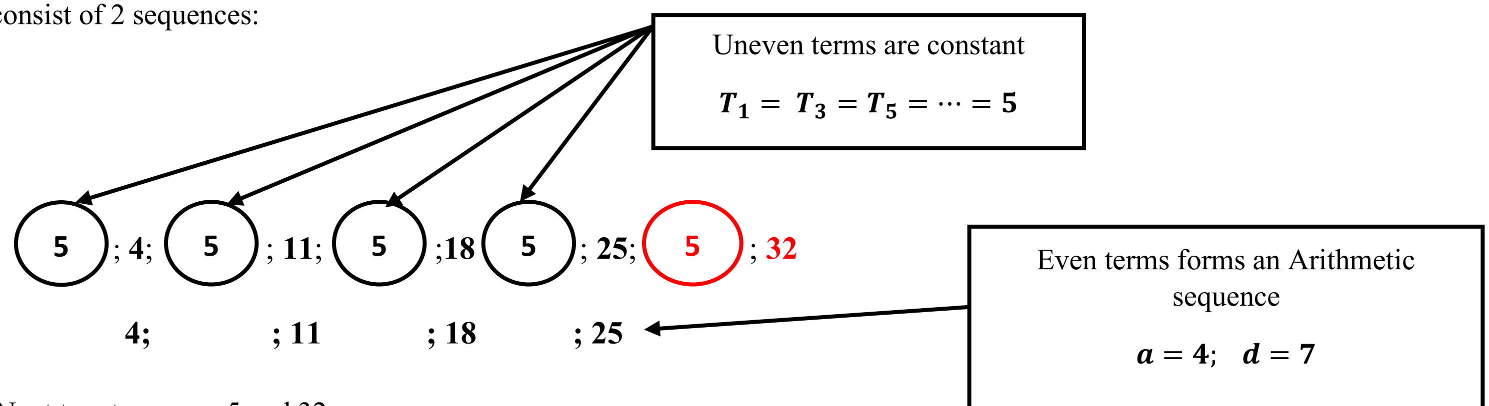
Consider the following sequence of numbers: **5; 4; 5; 11; 5; 18; 5; 25; …**

(a) Given that the pattern continues. Write down the next two terms of the sequence.

(b) Determine the sum of the first 50 terms of the sequence.

**Solution:**

Pattern consist of 2 sequences:




Next two terms are 5 and 32

(a) $S_{50} = S_{(25)uneven} + S_{(25)even}$

$= 25(5) + \frac{25}{2}[2(4) + (25-1)(7)]$

$= 125 + 2\ 200$

$= 2\ 325$

**Example 7**

Thabo trains to run the Two oceans marathon. He runs 4 km in the first week and increases his distance by 3 km each week after that.

(a) What distance did he run during the $11^{th}$ week?

(b) During which week did he run 25 km?

(c) What is the total distance he runs in the first 15 weeks?

(d) It is said that an athlete should run at least 900 km in preparation for the Two oceans marathon. How many weeks should Thabo train before he meets his goal?

**Solution:**

Method

1) Read the question and find the sequence first.
2) Determine what type of sequence Arithmetic or Geometric
3) Now apply the formula depending which sequence it is.

Arithmetic

$a = 4\,;\ d = 3;\ n = 11\quad T_{11} = ?$

(a) Sequence: 4; 7; 10; …

$T_n = a + (n-1)d$

$T_n = 4 + (n-1)(3) = 3n + 1$

$T_{11} = 3(11) + 1 = 34$

∴ He ran 34 km in week 11.

(b) $T_n = 25$

$3n + 1 = 25$

$\therefore 3n = 24$

$\therefore n = 8$

∴ He ran 25 km in week 8.

(c) $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{15} = \frac{15}{2}[2(4) + (15-1)(3)]$

$\therefore S_{15} = 375$

∴ He ran 375 km in the first 15 weeks.

(d) $S_n = 900$

$\frac{n}{2}[2(4) + (n-1)(3)] = 900$

$\therefore n[8 + 3n - 3] = 1800$

$\therefore 3n^2 + 5n - 1800 = 0$

$\therefore n = \frac{-5 \pm \sqrt{(5)^2 - 4(3)(-1800)}}{2(3)}$

$\therefore n = 23{,}68$ or $n = -25{,}34$

∴ He will have to train for 24 weeks.

**Example 8**

The price of a new car depreciates by 15 % per year. The new price of the car is R180 000.

(a) Write down a sequence showing the value of the car for the first 3 years.

(b) Determine a formula linking the value of the car (V) with its age (n)?

(c) The value of the car after 5 years.

$a = 180\ 000\,;\ r = 0{,}085$

$\mathrm{T_n} = V_n$ ?

**Solution to Example 8:**

(a)

| During the year | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Value of car | 180 000 | 180 000(0,085) | 180 000$(0,085)^2$ | 180 000$(0,085)^3$ |

(b)

$T_n = ar^{n-1}$

$T_n = 180\,000(0{,}085)^{n-1}$

$V_n = 180\,000(0{,}085)^{n-1}$

(c) After 5 years, thus 6 years

$V = 180\,000(0{,}085)^{n-1}$

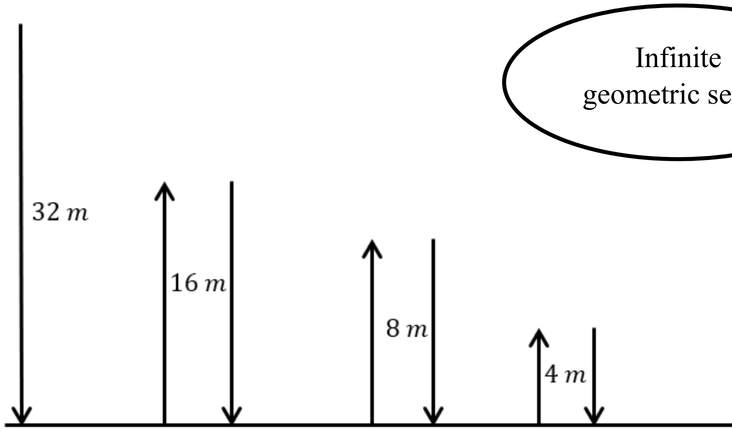
$V_6 = 180\,000(0{,}085)^{6-1}$

$V_6 = R\ 79\ 866{,}96$

**Example 9**

A ball is dropped from a height of 32 m and bounces continuously. With each successive bounce, the ball reaches a height that is 50% of the previous height. If this motion continues indefinitely, what is the total vertical distance travelled by the ball over its entire journey?

**Solution:** Draw a picture of the scenario whenever possible




Infinite geometric series

**Solution:**

From the sketch we can write the sequence as

Vertical Distance

$= 32 + 16 + 16 + 8 + 8 + 4 + 4 + \cdots$

$= 32 + 2(16) + 2(8) + 2(4) + \cdots$

$= 32 + 2(\mathbf{16} + \mathbf{8} + \mathbf{4} + \cdots)$

$= 32 + 2\left(\frac{16}{1-0{,}5}\right)$

$= 32 + 2(32)$

$= 96$

$S_\infty = \frac{a}{1-r}$

**∴ The total vertical distance is 96m**

| **CAN YOU?** | **Answers:** |
|---|---|
| 1. Thabo trains for the Spar cycle race. He cycles a distance of 50 km in the first week and increases his distance by 15 km each week after that.<br>(a) What distance did he cycle during the $15^{th}$ week?<br>(b) During which week did he run 185 km?<br>(c) What is the total distance he cycled over a 20 week period?<br>(d) Thabo first started cycling 15 weeks before the end of the year. His goal is to cycle 3000 km before the end of the year. Will he able to reach his goal?<br><br><br>2. A tree with an initial height of 2m is planted. In the first year it grows 1,5m. In the second year it grows 1,2m, in the third year 0,96m, in the fourth year 0,768m and so forth.<br>What is the maximum height that the tree can reach? | 1. (a) 260 km<br>(b) $n = 10$<br>(c) 3850 km<br>(d) $S_{15} = 2\ 325$ km,<br>$\therefore$ No, he will not reach his goal.<br>2. 9,5m |

## CONSOLIDATION

**Summary**

| | ARITHMETIC<br>$a;\ a+d;\ a+2d;\ \ldots$ | GEOMETRIC<br>$a;\ ar;\ ar^2;\ \ldots$ |
|---|---|---|
| **Definition** | **Constant difference:**<br>$d = T_2 - T_1 = T_3 - T_2$ | **Constant ratio:**<br>$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$ |
| **General Term** | $T_n = a + (n-1)d$ | $T_n = ar^{n-1}$ |
| **Sum** | $S_n = \frac{n}{2}[2a + (n-1)d]$<br>$S_n = \frac{n}{2}[a + l]$ | $S_n = \frac{a(r^n - 1)}{r-1}$<br>$S_n = \frac{a(1-r^n)}{1-r}$ |
| **Sum to infinity** | **Not applicable** | $S_\infty = \frac{a}{1-r}$ |

## ACTIVITIES/ASSESSMENT

| **Textbook** | **Mind Action Series** | **Everything Maths Siyavula** | **Classroom Mathematics** | **Via Afrika Mathematics** |
|---|---|---|---|---|
| **Sum to infinity** | Ex: 6<br>Page:19 | Ex: 1.4- 1.61<br>Page: 14-18 | Ex: 1.9<br>Page: 31 | Ex:5<br>Page: 29 |
| **Applications** | Ex: 10<br>Page:33 | Ex: 1.9<br>Page 35 | Ex: 1.11<br>Page: 35 | Ex: 6<br>Page:31 |