Western Cape Government
Education

Directorate: Curriculum FET

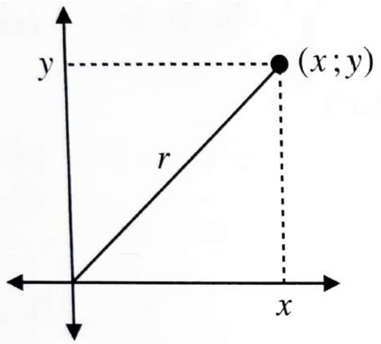
SUBJECT and GRADE	Mathematics	
TERM 1	Week 7	
TOPIC	Trigonometry – Compound and Double angles	
AIMS OF LESSON	Compound angles identities Double angle identities	
RESOURCES	***Paper based resources***	***Digital resources***
	Siyavula; Your textbook	https://www.youtube.com/watch?v=VZt4wXBo1PA

INTRODUCTION

Let's go back and review the basic concepts.

OVERVIEW OF BASIC CONCEPTS

DEFINITIONS AND PYTHAGORUS:



$$x^2 + y^2 = r^2$$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

REDUCTION AND CO-FUNCTION FORMULAE:

Second Quadrant:

- **sine function is positive**
- $\sin(180° - \theta) = \sin\theta$
- $\cos(180° - \theta) = -\cos\theta$
- $\tan(180° - \theta) = -\tan\theta$
- $\sin(90° + \theta) = \cos\theta$
- $\cos(90° + \theta) = -\sin\theta$

First Quadrant:

- **all functions are positive**
- $\sin(360° + \theta) = \sin\theta$
- $\cos(360° + \theta) = \cos\theta$
- $\tan(360° + \theta) = \tan\theta$
- $\sin(90° - \theta) = \cos\theta$
- $\cos(90° - \theta) = \sin\theta$

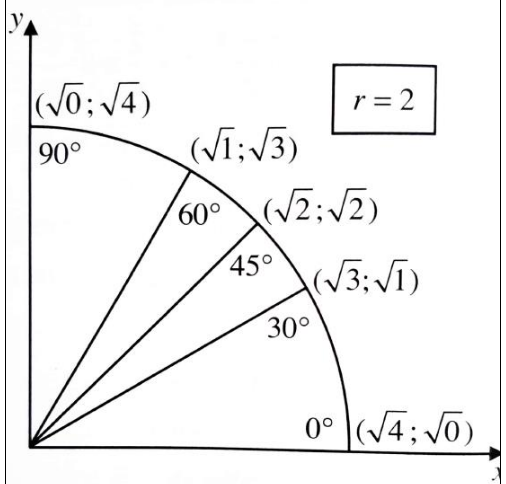
Third Quadrant:

- **tangent function is positive**
- $\sin(180° + \theta) = -\sin\theta$
- $\cos(180° + \theta) = -\cos\theta$
- $\tan(180° + \theta) = \tan\theta$

Fourth Quadrant:

- **cosine function is positive**
- $\sin(360° - \theta) = -\sin\theta$
- $\cos(360° - \theta) = \cos\theta$
- $\tan(360° - \theta) = -\tan\theta$

SPECIAL ANGLES:



IDENTITIES:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta = 1 - \cos^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta$$

DISTANCE FORMULA:

$$AB^2 = (x_A - x_B)^2 + (y_A - y_B)^2$$
$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

COSINE rule:

In ΔABC:

$$AB^2 = AC^2 + BC^2 - 2(AC)(BC)\cos\hat{C}$$

CONCEPTS AND SKILLS

COMPOUND ANGLES: A compound angle is an angle formed by adding and subtracting two angles, for example $\hat{A} + \hat{B}$, $\alpha - \beta$, $\theta + 30°$, $x - 60°$ etc.

INVESTIGATION: Joey is investigating the following question: $\cos(180° - 120°)$

Joey's solution:

$$\cos(180° - 120°) = \cos 180° - \cos 120°$$
$$= -1 - (\cos(90° + 30°)$$
$$= -1 + \sin 30°$$
$$= -1 + \frac{1}{2}$$
$$= -\frac{1}{2}$$

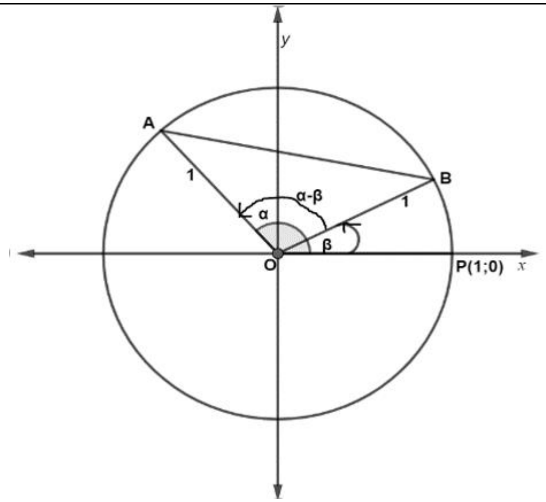
1. Consider Joey's solution and determine why it is incorrect.
2. Use a calculator to check that Joey's answer is wrong.
3. Describe in words the mistake(s) in his solution.
4. Is the following statement true or false?

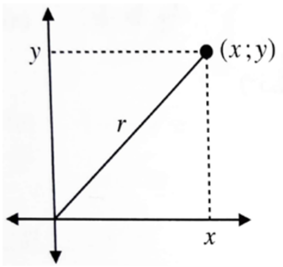
"A trigonometric ratio can be distributed to the angles that lie within the brackets."

From the investigation above, we know that $\cos(\alpha - \beta) \neq \cos\alpha - \cos\beta$. **It is wrong to apply the distributive law to the trigonometric ratios of compound angles**.

Derivation of $\cos(\alpha - \beta)$

Using the **distance formula** and the **cosine rule**, we can derive the following identity for compound angles: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$



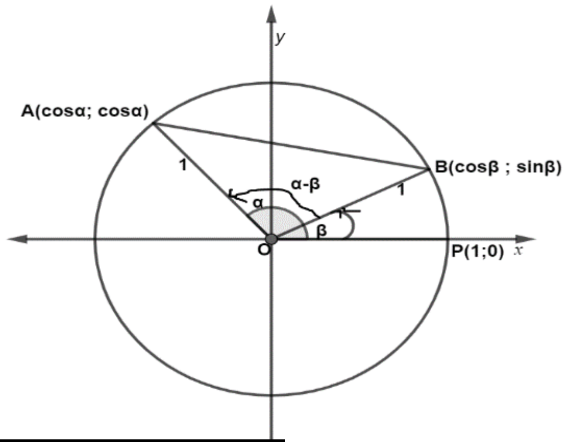


$\frac{x}{r} = \cos\theta$

$x = r\cos\theta$

$\frac{y}{r} = \sin\theta$

$y = r\sin\theta$



$AB^2 = (x_A - x_B)^2 + (y_A - y_B)^2$ **(distance formula)**

$= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$

$= 1 - 2\cos\alpha\cos\beta + 1 - 2\sin\alpha\sin\beta$

$= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

In ΔABO : applying the **cosine formula**

$AB^2 = AO^2 + BO^2 - 2(AO)(BO)\cos(\alpha - \beta)$

$= 1 + 1 - 2\cos(\alpha - \beta)$

$= 2 - 2\cos(\alpha - \beta)$

Hence: $2 - 2\cos(\alpha - \beta) = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$-2\cos(\alpha - \beta) = -2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

<u>Compound Angles: Identities</u>

$\mathbf{\cos(A - B) = \cos A\cos B + \sin A\sin B}$

$\mathbf{\cos(A + B) = \cos A\cos B - \sin A\sin B}$

$\mathbf{\sin(A + B) = \sin A\cos B + \cos A\sin B}$

$\mathbf{\sin(A - B) = \sin A\cos B - \cos A\sin B}$

Deriving the other compound angle identities:

$$\begin{aligned}\mathbf{cos(A+B)} &= \mathbf{cos(A-(-B))}\\ &= \mathbf{cosA\ cos(-B) + sinA\ sin(-B)}\\ &= \mathbf{cos\,A\ cosB - sinA\ sinB}\end{aligned}$$

$$\begin{aligned}\mathbf{sin(A-B)} &= \mathbf{sin(A+(-B))}\\ &= \mathbf{sinA\ cos(-B) + cosA\ sin(-B)}\\ &= \mathbf{sinA\ cosB - cosA\ sinB}\end{aligned}$$

$$\begin{aligned}\mathbf{sin(A+B)} &= \mathbf{cos(90^\circ-(A+B))}\\ &= \mathbf{cos((90^\circ-A)-B)}\\ &= \mathbf{cos(90^\circ-A)\ cosB + sin(90^\circ-A)\ sinB}\\ &= \mathbf{sinA\ cosB + cosA\ sinB}\end{aligned}$$

These formulae can be used to **expand** a trigonometric ratio of a compound angle into an expression consisting of trigonometric ratios of the individual angles.

EXAMPLE 1: Expand the following and simplify if possible:

(a) $\sin(x+y) = \mathbf{\sin x \cos y + \cos x \sin y}$

(b) $\cos(\alpha+10^\circ) = \mathbf{\cos\alpha\cos 10^\circ - \sin\alpha\sin 10^\circ}$

Compound angle identity

(c) $\cos(x+60^\circ) = \mathbf{\cos x\ \cos 60^\circ - \sin x\ \sin 60^\circ}$ — Compound angle identity

$= \mathbf{\cos x.\left(\frac{1}{2}\right) + \sin x.\left(\frac{\sqrt{3}}{2}\right)}$ — Special Angles

$= \mathbf{\frac{\cos x+\sqrt{3}\ \sin x}{2}}$ — Simplify

(d) $\sin(3x-45^\circ) = \mathbf{\sin 3x\cos 45^\circ + \cos 3x \sin 45^\circ}$ — Compound angle identity

$= \mathbf{\sin 3x.\left(\frac{\sqrt{2}}{2}\right) + \cos 3x.\left(\frac{\sqrt{2}}{2}\right)}$ — Special Angles

$= \mathbf{\frac{\sqrt{2}\sin 3x - \sqrt{2}\ \cos 3x}{2}}$

The compound angle formulae can also be used to **contract** certain trigonometric expressions into a **single trigonometric ratio**:

EXAMPLE 2: Write the following as a single trigonometric ratio:

(a) $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \boldsymbol{\cos(\alpha - \beta)}$

(b) $\sin x\cos 40° + \cos x\sin 40° = \boldsymbol{\sin(x + 40°)}$

(c) $\cos 2x\sin 3x - \sin 2x\cos 3x$ — rearrange terms in identity form

$= \sin 3x\cos 2x - \cos 3x\sin 2x$ — Compound angle identity

$= \sin(3x - 2x)$

$= \sin x$

EXAMPLE 3: Simplify: $\sin(x + 30°) - \sin(x - 30°)$

Solution:

$$\begin{aligned}\sin(x+30°) - \sin(x-30°) &= \boldsymbol{\sin x\cos 30°} + \boldsymbol{\cos x\sin 30°} - (\boldsymbol{\sin x\cos 30°} - \boldsymbol{\cos x\sin 30°})\\ &= \sin x\cos 30° + \boldsymbol{\cos x\sin 30°} - \sin x\cos 30° + \boldsymbol{\cos x\sin 30°}\\ &= 2\cos x\,\boldsymbol{\sin 30°}\\ &= 2\cos x\left(\boldsymbol{\tfrac{1}{2}}\right)\\ &= \cos x\end{aligned}$$

EXAMPLE 4: Calculate the values of the following without the use of a calculator:

(a) $\cos 75° = \boldsymbol{\cos(30° + 45°)}$ — Write i.t.o. special angles

$= \cos 30°\cos 45° - \sin 30°\sin 45°$ — Compound angle identity

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$ — Simplify

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

CAN YOU:

1) Expand the following.

a) $\cos(\alpha + 70°) =$
b) $\sin(x - 45°) =$

2) Write the following as a single trigonometric ratio.

a) $\cos 14°\sin 6° + \cos 6°\sin 14°$
b) $\cos\theta\cos 4\theta - \sin\theta\sin 4\theta$

(b) $\sin 70° \sin 10° + \cos 10° \cos 70°$

$= \cos 70° \cos 10° + \sin 70° \sin 10°$

$= \cos(70° - 10°)$

$= \cos 60°$

$= \frac{1}{2}$

rearrange terms in identity form

Compound angle identity

3) Simplify the following without the use of calculator.

a) $\cos 37° \cos 23° - \sin 37° \sin 23°$

b) $\cos 15°$

c) $\sin 80° \sin 55° - \sin 10° \cos 67°$

<u>DOUBLE ANGLES</u>: A double angle is an formed by adding an angle to itself, for example $\hat{\mathbf{A}} + \hat{\mathbf{A}} = \mathbf{2\hat{A}}$; $\boldsymbol{\alpha + \alpha = 2\alpha}$; $\mathbf{10° + 10° = 20°}$ e.t.c.

$\sin 2A = \sin(A + A)$

$= \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A$

$$\boxed{\mathbf{\sin 2A = 2 \sin A \cos A}}$$

$\cos 2A = \cos(A + A)$

$= \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$

$$\boxed{\mathbf{\cos 2A = \cos^2 A - \sin^2 A}}$$

By using the identity

$$\boxed{\begin{gathered}\mathbf{\sin^2 A + \cos^2 A = 1} \\ \mathbf{\cos^2 A = 1 - \sin^2 A} \\ \mathbf{\sin^2 A = 1 - \cos^2 A}\end{gathered}}$$

, we can rewrite the formula **cos 2A** also as

$\cos 2A = \mathbf{\cos^2 A} - \sin^2 A$

$= \mathbf{1 - \sin^2 A} - \sin^2 A$

$= 1 - 2\sin^2 A$

$$\boxed{\mathbf{\cos 2A = 1 - 2\sin^2 A}}$$

$\cos 2A = \cos^2 A - \mathbf{\sin^2 A}$

$= \cos^2 A - (\mathbf{1 - \cos^2 A})$

$= 2\cos^2 A - 1$

$$\boxed{\mathbf{\cos 2A = 2\cos^2 A - 1}}$$

EXAMPLE 5: Simplify the following:

a) $\cos 2x + 2\sin^2 x$

$= 1 - 2\sin^2 x + 2\sin^2 x$
$= 1$

b) $\frac{\sin 2\theta}{\sin\theta}$

$= \frac{2\sin\theta\cos\theta}{\sin\theta}$

$= 2\cos\theta$

c) $\frac{\cos 2A + 1}{2\cos A}$

$= \frac{2\cos^2 A - 1 + 1}{2\cos A}$

$= \frac{2\cos^2 A}{2\cos A} \quad = \cos A$

EXAMPLE 6: Calculate the following without the use of a calculator.

a) $2\sin 15°\cos 15°$
$= \sin 2(15)$
$= \sin 30°$
$= \frac{1}{2}$

b) $\sin 22{,}5°\cos 22{,}5°$
$= \frac{1}{2}\,(2\sin 22{,}5°\cos 22{,}5°)$
$= \frac{1}{2}\sin(2 \times 22{,}5)$
$= \frac{1}{2}\sin(45°)$
$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)$
$= \frac{\sqrt{2}}{4}$

c) $2\cos^2 15°$
$= 2\cos^2 15° - 1 + 1$
$= \cos(2 \times 15°) + 1$
$= \cos 30° + 1$
$= \frac{\sqrt{3}}{2} + 1$
$= \frac{\sqrt{3}+2}{2}$

EXAMPLE 7: If $\mathbf{5\tan\alpha - 12 = 0}$ with $\boldsymbol{\alpha \in [180°; 360]}$ and $\mathbf{5\cos\beta = -3}$ with $\mathbf{\sin\beta > 0}$ **,** determine the value of the following trigonometric ratios without the use of a calculator and with the aid of a diagram .

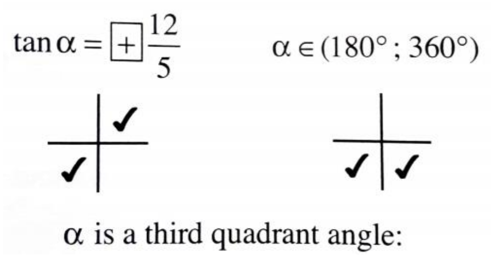
a) $\sin(\alpha + \beta)$ b) $\sin(\alpha - \beta)$ c) $\cos(\alpha + \beta)$ d) $\sin 2\beta$

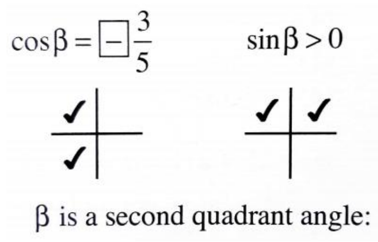
Solution:

a) $\sin(\alpha + \beta)$
$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$= \left(\frac{-12}{13}\right)\left(\frac{-3}{5}\right) + \left(\frac{-5}{13}\right)\left(\frac{4}{5}\right) = \frac{16}{65}$

b) $\sin(\alpha - \beta)$
$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$= \left(\frac{-12}{13}\right)\left(\frac{-3}{5}\right) - \left(\frac{-5}{13}\right)\left(\frac{4}{5}\right) = \frac{56}{65}$

Rough work





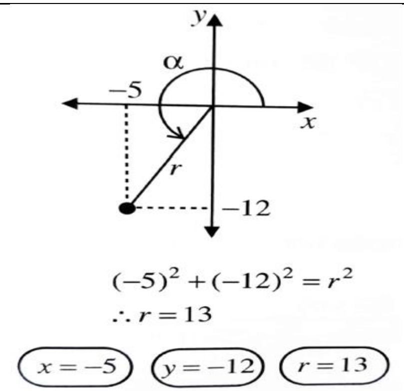
c) $\cos(\alpha + \beta)$

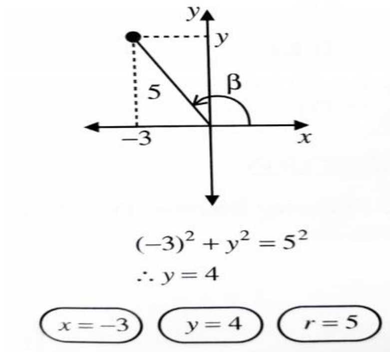
$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(\frac{-5}{13}\right)\left(\frac{-3}{5}\right) - \left(\frac{-5}{13}\right)\left(\frac{4}{5}\right) = \frac{63}{65}$

d) $\sin 2\beta$

$= 2\sin\beta\cos\beta$





PROVING IDENTITIES: When proving identities, we always work with the two sides (LHS and RHS) separately and simplify or manipulate the two sides till they equal the same expression.

HINT: (1) Start simplifying the more complex side of the identity. Replace, $\boldsymbol{\tan\theta}$ by $\boldsymbol{\frac{\sin\theta}{\cos\theta}}$.

(2) Replace the double angles or compound angles by single or reduced angles where posible.

(3) Thus **replace,** $\boldsymbol{\sin 2\theta}$ by $\boldsymbol{2\sin\theta\cos\theta}$**.**

(4) $\boldsymbol{\cos 2x}$ have 3 identities you can choose from. It helps if **you look forward to what you must prove and then choose the identity that will support the simplification of the expression, to support you getting LHS = RHS.**

EXAMPLE 8:

Prove that $\frac{\sin 2x}{1+\cos 2x} = \tan x$

Solution:

LHS = $\frac{\sin 2x}{1+\cos 2x}$

$= \frac{2\sin x\cos x}{1+(2\cos^2 x - 1)}$

$= \frac{2\sin x\cos x}{2\cos^2 x}$

$= \frac{\sin x}{\cos x} = \tan x$

$= RHS$

EXAMPLE 9:

Prove: $\sin 3x = 3\sin x - 4\sin^3 x$

Solution:

LHS $= \sin 3x$

$= \sin(2x + x)$

$= \sin 2x\cos x + \cos 2x\sin x$

$= 2\sin x\cos x\cos x + (1 - 2\sin^2 x)\sin x$

$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x$

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

$= RHS$

EXAMPLE 10:

Prove: $\frac{1-\cos 2x}{\sin x} = \frac{\sin 2x}{\cos x}$

Solution:

LHS= $\frac{1-\cos 2x}{\sin x}$

$= \frac{1-(1-2\sin^2 x)}{\sin x}$

$= \frac{2\sin^2 x}{\sin x} = \boldsymbol{2\sin x}$

RHS $= \frac{\sin 2x}{\cos x} = \frac{2\sin x\cos x}{\cos x}$

$= \boldsymbol{2\sin x}$

CAN YOU:

1.) Prove the following identities:

a) $(\sin x + \cos x)^2 = 1 + \sin 2x$

b) $\tan\theta + \frac{cos\theta}{\sin\theta} = \frac{2}{\sin 2\theta}$

c) $\frac{\sin\beta - \sin 2\beta}{\cos\beta - \cos 2\beta - 1} = \tan\beta$

2.) If $25\sin\alpha - 7 = 0$, with $90° \leq \alpha \leq 270°$, and $5\cos\beta = 4$, with $180° \leq \beta \leq 360°$,
Determine with the aid of a sketch the value of :

a) $\tan\alpha$
b) $\sin 2\beta$
c) $\tan 2\beta$
d) $\cos(\alpha - \beta)$
e) $\sin(\alpha - \beta)$

ACTIVITIES/ ASSESSMENT

Clever	Mind Action Series	Classroom Mathematics	Via Afrika Mathematics
Ex: 5.1 & 5.5 Pg.: 111 & 118	Ex: 1 & 8 Pg.: 143 & 164	Ex: 5.5 & 5.8 Pg.: 138 & 144	Ex:5.2 & 5.4 Pg.: 110 & 119

CONSOLIDATION

Compound Angles: Identities

$\mathbf{\cos(A - B) = \cos A \cos B + \sin A \sin B}$

$\mathbf{\cos(A + B) = \cos A \cos B - \sin A \sin B}$

$\mathbf{\sin(A + B) = \sin A \cos B + \cos A \sin B}$

$\mathbf{\sin(A - B) = \sin A \cos B - \cos A \sin B}$

DOUBLE ANGLES

$\mathbf{\sin 2A = 2\sin A \cos A}$
$\mathbf{\cos 2A = 1 - 2\sin^2 A}$
$\mathbf{\cos 2A = 2\cos^2 A - 1}$
$\mathbf{\cos 2A = \cos^2 A - \sin^2 A}$