Western Cape Government
Education

Directorate: Curriculum FET

SUBJECT and GRADE	Mathematics– Grade 10	
TERM 2	*Week 4*	
TOPIC	Functions and Graphs - Hyperbola	
AIMS of LESSON	1. To introduce you to the shape and the equation of a hyperbola. 2. Show you how the parameters “a” and “q”, impact on the graph of the hyperbola. 3. Draw a sketch of the hyperbola and find the equation when a sketch is given.	
RESOURCES	***Paper based resources***	***Digital resources***
	Please go to the Functions section in your Mathematics Textbook. Topic – The hyperbola	https://youtu.be/b-j4d1TuQpY https://youtu.be/QmtxAh2fDCg

INTRODUCTION: Dear Learner, in the previous lesson you have learnt about the shape of the parabola, how to sketch it and find the equation of the quadratic function of the form, $f(x) = ax^2 + q$. You should now know how to calculate the y- coordinate of a point on a function when $x - coordinate\ is\ given$ and be able to plot these points on a cartesian plane.

CONCEPTS AND SKILLS: In this lesson we will explore the function with the equation, $\boldsymbol{f(x) = \frac{a}{x} + q}$, *the **Hyperbolic function***. To start, we first need to explore the shape of this graph. We will do this by doing example 1 below.

Example 1:

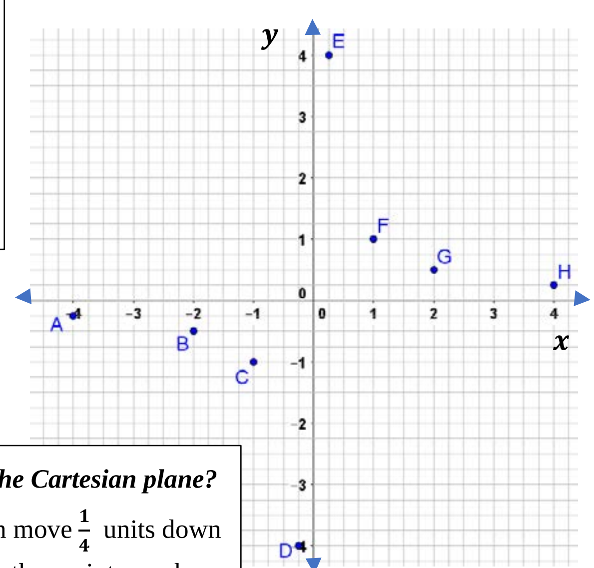
Consider the function $f(x) = \frac{1}{x}$

x	-4	-2	-1	$-\frac{1}{4}$	0	$\frac{1}{4}$	1	2	4
$f(x)$									

a) Complete the above table.
b) Check your table with the one under **solution**.
c) The points have been plotted on the given Cartesian plane to the right, try on your own, to ensure you are able to plot points.

Solution:

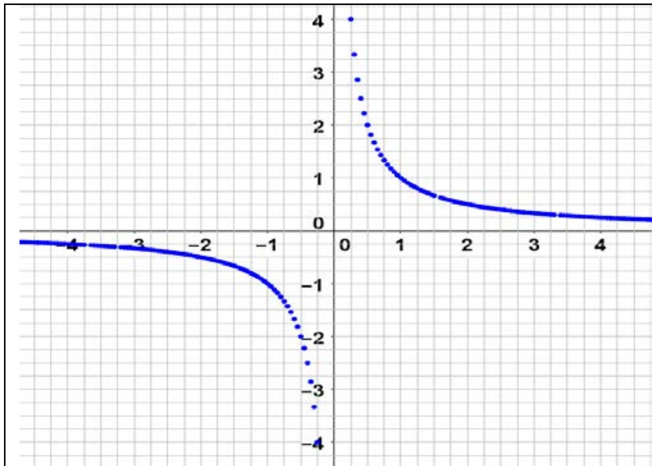
x	-4	-2	-1	$-\frac{1}{4}$	0	$\frac{1}{4}$	1	2	4
$f(x)$	$-\frac{1}{4}$	$-\frac{1}{2}$	1-	-4		4	1	$\frac{1}{2}$	$\frac{1}{4}$



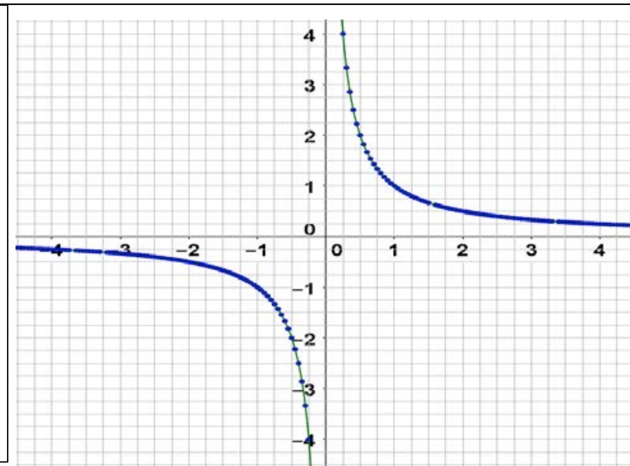
Note: $f(x)$ is the y-coordinate of each point. The points are named so that it becomes easy to refer to the points. So the following is obtained from the table:
A(-4;$-\frac{1}{4}$) , B(-2; $-\frac{1}{2}$), C(-1;-1) , D($-\frac{1}{4}$; - 4), E($\frac{1}{4}$; 4), F(1; 1), G(2; $\frac{1}{2}$) and H(4; $\frac{1}{4}$)

How to plot A(-4;$-\frac{1}{4}$) on the Cartesian plane?

Go to -4 on the x- axis, then move $\frac{1}{4}$ units down along the y-axis. All the other points can be plotted in the same way.



If you were to take more x- values and calculate the corresponding y-values for the function, $f(x) = \frac{1}{x}$ and then plot the points, you will find the image in the sketch on the left-hand side appearing. I am sure you will agree if more and more points were determined and plotted the graph on the right-hand side will eventually emerge. **Notice** this graph does not have an **x- or y - intercept**.



Let's investigate the effect of the value of "***a***" **and** "***q***" on the shape of different hyperbolas:

Example 2: Consider the function:

$$k(x) = -\frac{1}{x}$$

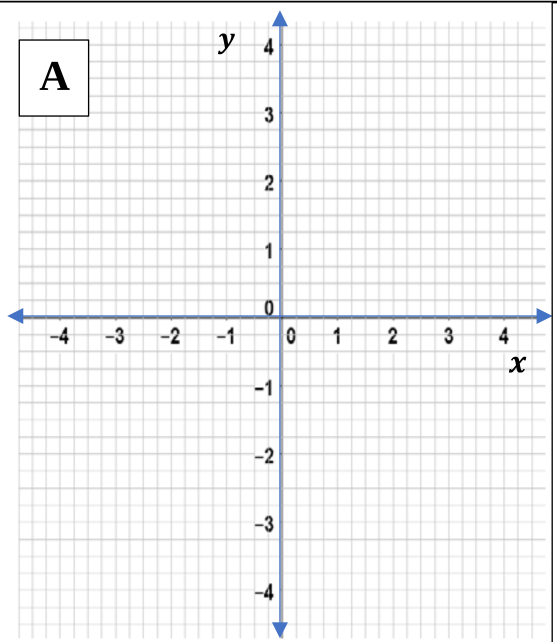
x	-4	-2	-1	$-\frac{1}{4}$	$\frac{1}{4}$	1	2	4
$k(x)$			1			-1		

a) Complete the table above.

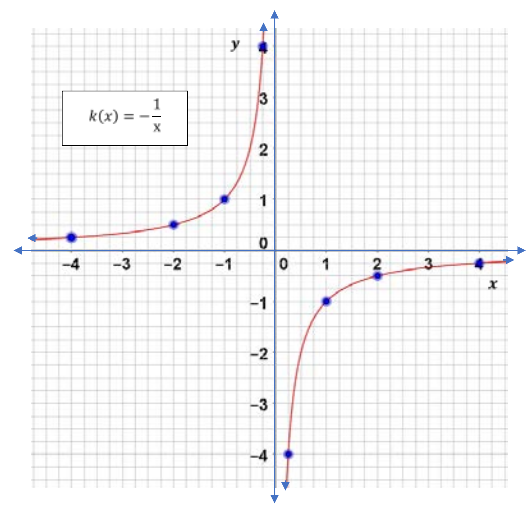
b) Plot the points of $k(x)$ on the Cartesian plane A.

c) Join the points plotted in b).

d) What do you observe about this graph compared to the above graph of $f(x) = \frac{1}{x}$?



Solution:

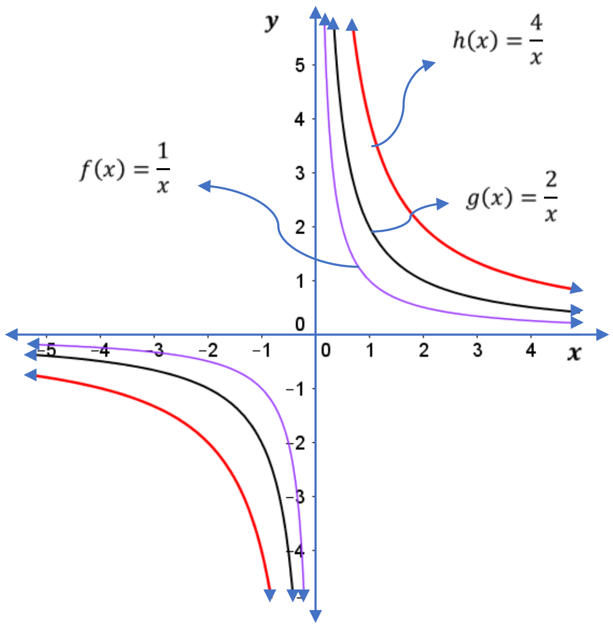


Can you:

Sketch the graph of, $j(x) = -\frac{6}{x}$

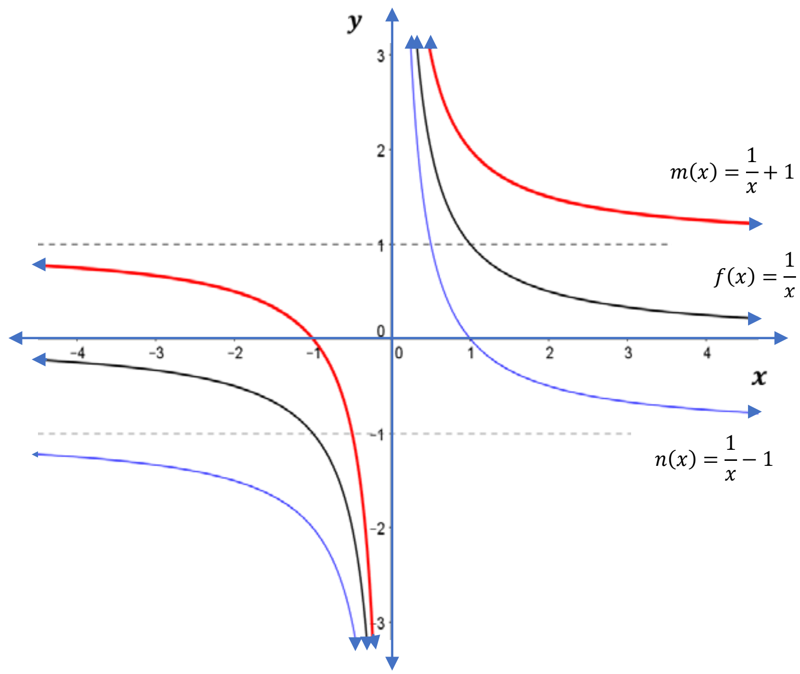
In the same way as we have sketched the function $f(x)=\frac{1}{x}$ and $k(x)=-\frac{1}{x}$ above, we can sketch any other hyperbola, by choosing suitable x-values calculating their corresponding y-values, plotting the points and then joining the points.

Using a computer,
$f(x)=\frac{1}{x}$, $g(x)=\frac{2}{x}$ and $h(x)=\frac{4}{x}$
were sketched on the same system of axes. See the graph below.



Notice that as the number in the numerator "a" gets larger, the branches of the hyperbolas are stretched vertically away from the x-axis. The branches of the graph of $h(x)=\frac{4}{x}$ are stretched further away from the x-axis compared to the graph of $g(x)=\frac{2}{x}$.

Using a computer,
$f(x)=\frac{1}{x}$, $m(x)=\frac{1}{x}+1$, and $n(x)=\frac{1}{x}-1$
were sketched on the same system of axes. See the graph below.



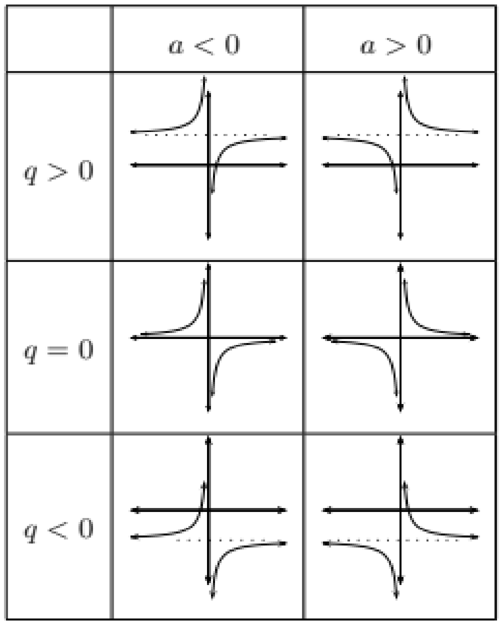
Notice that the graph of $\boldsymbol{m(x)=\frac{1}{x}+1}$ is the graph of f shifted 1 unit up.

The graph of $n(x)=\frac{1}{x}-1$, is the graph of f shifted 1 unit down.

Characteristics of Hyperbola with Equation: $y = \frac{a}{x} + q,$ Where $x \neq 0$ and $y \neq q$

Asymptotes: These are lines that the graph approaches but never touches or intersect.

Below are rough sketches for the hyperbola for various value(s) of "a" and "q"



Characteristics

- **Asymptotes:** The Hyperbola has two asymptotes.
  - $x = 0$ (this is the y-axes) and $y = q$ are the equations of the two asymptotes
- **Shape and Quadrant of the hyperbola**
  - If a is more than 0, the curves will be in the 1st and 3rd quadrants.
  - If a is less than 0, the curve will be in the 2nd and 4th quadrant
- q determines the number of units the hyperbola is shifted above or below the x – axes.
  - If q is more than 0, then the hyperbola will be shifted q units above the x axis.
  - If q is less than 0, then the hyperbola will be shifted q units below the x axis.
  - If q = 0 then, the x- axes is the second asymptote.
- Lines of Symmetry: The hyperbola has two lines of Symmetry.
  - $y = x + q$ and $y = -x + q$
- **Domain:** $x \in R$, $x \neq 0$
- **Range:** $\boldsymbol{y \in R}$, $\boldsymbol{y \neq q}$

This is because the equation of hyperbola is: $y = \frac{a}{x} + q$, if $x = 0$ the function is not defined

Sketch the Hyperbola:

You are expected to sketch the Hyperbola by using the characteristics of the Hyperbola, and not by using point by point plotting.

To sketch the hyperbola:

1. Establish the shape of the graph by identifying if the equation is of the form, $y = \frac{a}{x} + q$, if it is, you will know that it is a hyperbola. The sign of "a", will indicate in which quadrants the hyperbola is, if $a > 0$ then the hyperbola is in the 1st and 3rd quadrant, if $a < 0$ then hyperbola is in the 2nd and 4th quadrant.
2. Write down the asymptotes. That is $x = 0$, is the vertical asymptote(that is the y-axes) and $y = q$ is the horizontal asymptote.
3. Determine the x- intercepts. Some hyperbolas do not have an x-intercept. If a hyperbola does not have an x- intercept, determine the coordinates of any other point on the graph and plot this on the sketch.

Sketching Hyperbolas

Example 3

1.1 Sketch the graph of, $f(x) = -\frac{4}{x}$

1.2 Write down the domain.

1.3 Write down the range.

Solutions:

1.1 a) Shape is hyperbola $f(x) = \frac{a}{x}$

a $<$ 0, therefore sketch in 2nd & 4th quadrant.

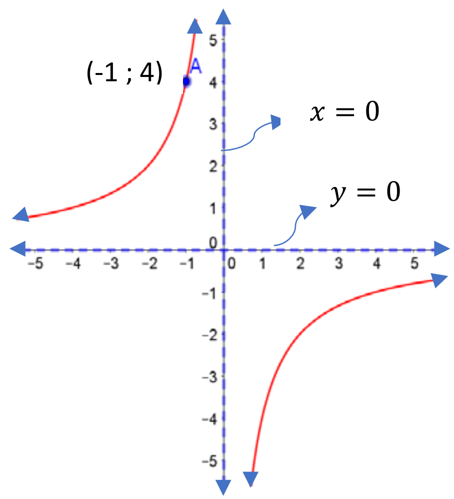
b) Asymptotes: $x = 0$ & $y = 0$

c) As y=0 is an asymptote, there are no x-intercepts.

1.2 Domain: $x \in R$, $x \neq 0$

1.3 Range: $y \in R$, $y \neq 0$

Sketch of $f(x) = -\frac{4}{x}$



Steps to sketching the graph

1. Establish the form of equation to determine, whether the graph is a parabola, hyperbola or exponential.
   $a < 0$ ∴ Hyperbola in 2nd & 4th quadrant
2. Write down the asymptotes.
3. Determine the x-intercept. If no x-intercept determine another point on the graph.

Use the above to, sketch and label the graph.

Can you:

Follow the steps and Sketch the graph of:

$f(x) = \frac{6}{x}$

Example 4

Sketch the following graph showing all intercepts with the axes and asymptotes: $g(x) = -\frac{4}{x} + 1$

Solutions

1. Standard form (yes)
2. Shape: $a < 0$ (2nd and 4th quadrant)
3. x-intercepts: let $y = 0$

$0 = -\frac{4}{x} + 1$

$0 = -4 + x$ (multiplying by LCD which is x)

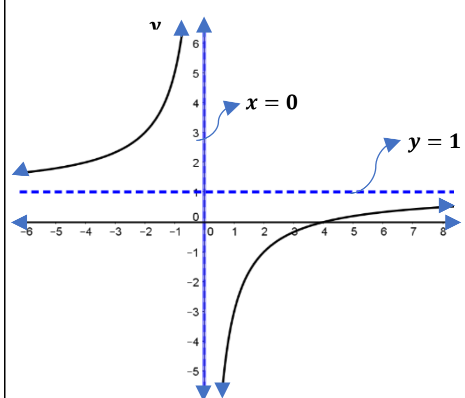
$x = 4$

Therefore (4 ; 0) is the x-intercepts

Domain: $x \in R$, $x \neq 0$

Range: $y \in R$, $y \neq 1$

Sketch of $\boldsymbol{y = -\frac{4}{x} + 1}$



Can you:

Sketch the graph of $y = \frac{3}{x} - 2$

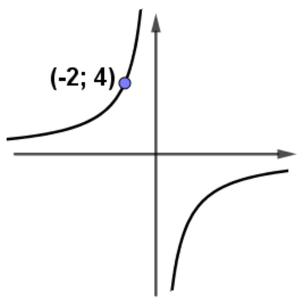
showing all the intercepts with the axes and asymptotes? Also state the domain and range.

Finding the equation of the hyperbola. There are two methods to use, depending on the information you have been given.

Example 5

Method 1:

Given the horizontal asymptote and or any other point on the graph.



Step 1: Use the general equation

Step 2: The horizontal asymptote gives you the value of q

Step 3: Substitute the coordinates of the point you have been given to find a

Solution to Example 5

$y = \frac{a}{x} + q$

$q = 0$ therefore $y = \frac{a}{x}$

substitute (-2; 4) in $y = \frac{a}{x}$

$\therefore 4 = \frac{a}{-2}$

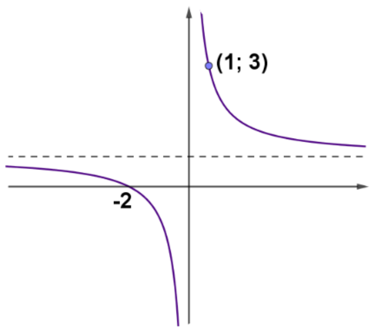
$\therefore a = -8$

$\therefore y = -\frac{8}{x}$

Example 6

Method 2:

Given any two points on the graph



Step 1: Use the general equation

Step 2: Substitute the coordinates of the first point into the equation to give you equation ①.

Step 3: Substitute the coordinates of the second point into the equation to give you equation ②.

Step 4: Solve equation ① and ② simultaneously to find the values of a and q

Solution to Example 6

$$y = \frac{a}{x} + q$$

Substitute (1; 3) into $y = \frac{a}{x} + q$

$3 = \frac{a}{1} + q$

$\therefore 3 = a + q$ ....①

Substitute (-2; 0) into $y = \frac{a}{x} + q$

$0 = \frac{a}{-2} + q$

$\therefore \quad 0 = a - 2q$ ....②

Subtract equation ② from ①

$3 = 3q$

$\therefore q = 1$

Substitute $q = 1$ into equation ①

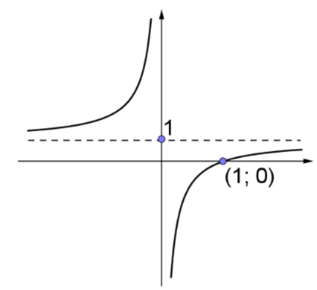
$3 = a + 1$

$\therefore \quad a = 2$

$\therefore y = \frac{2}{x} + 1$

Can you:

<u>Answer the following questions on the graph</u> below.



1. Determine the equation of the function.
2. What is the equation of the horizontal asymptote?.
3 What is the range of the function.
4 Determine the equation of the graph if the graph in the sketch above is shifted down by 3 units.

Answers:

1. $y = -\frac{1}{x} + 1$
2. $y = 1$
3. Range: $\boldsymbol{y} \in R$, $y \neq \mathbf{1}$
4. $y = -\frac{1}{x} - 2$

<table>
<tr><td>ACTIVITIES/ASSESSMENT</td><td><table><tr><td>Everything Maths Siyavula</td><td>Classroom Mathematics</td><td>Mind Action series</td></tr><tr><td>Ex.5- 4
Pg. 144 - 145</td><td>Ex 8.3 ; 8.4; 8.5
Pg. 170 - 178</td><td>Ex 3
Pg. 124</td></tr></table></td></tr>
<tr><td>CONSOLIDATION</td><td>You should be able to:
1. Rewrite equation in standard form.
2. Determine the orientation of the curves on the axes using a.
3. Determine the position of the curves in relation to the x axis using q.
4. Determine the asymptotes.
5. Using the asymptotes, a, q and the x-intercept, draw and label the graph.
6. Determine the Range and Domain.</td></tr>
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