SUBJECT and GRADE	MATHEMATICS GR 10
TERM 2	Week 5
TOPIC	FUNCTIONS – THE EXPONENTIAL FUNCTION $y = ab^x + q$, $b > 0$ and $b \neq 1$
AIMS OF LESSON	To:

- Introduce you to the shape, the standard form of the equation of an exponential function, and the impact of "a" and "q".
- Sketch the exponential function using the characteristics of it.
- Determine the Domain and Range for the exponential function.
- Determine the equation of the exponential function when the sketch is given.

RESOURCES	***Paper based resources***	***Digital resources***
	Please go to the Chapter on Functions and then to the section on the Exponential function in your Mathematics Textbook.	https://www.youtube.com/watch?v=b-ugmG3UlAc https://www.youtube.com/watch?v=DASfP8KAyvs https://www.youtube.com/watch?v=tQdXVvcKyp8

INTRODUCTION:

- In the previous lessons on Functions and Graphs you learned about the function notation, $f(x)$ and we looked at the graphs of the Parabola and the Hyperbola and their characteristics. In this lesson we will be focusing on the exponential function.
- We call the basic functions for the parabola: $f(x) = x^2$ and the hyperbola: $f(x) = \frac{1}{x}$ the "mother functions". In the same way the mother function for the exponential function is: $f(x) = b^x$. We can use this functions to draw any other derivations of the parabola, hyperbola and exponential function, given in that form, by performing some transformation on the "mother function".
- We will investigate the form and other characteristics of the exponential function through point-by-point plotting of some graphs.

Lesson 1a | Characteristics of exponential function through point-by-point plotting

Example 1:

Consider the function $f(x) = 2^x$. Complete the table below and plot the points on the Cartesian plane.

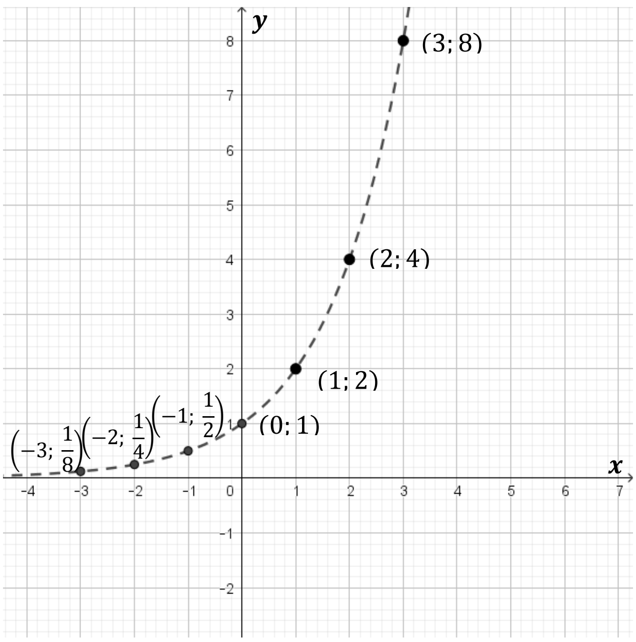
x	-3	-2	-1	0	1	2	3
$f(x)$							

What do we see:

- If we join the points, we see a graph that is **increasing**: as the x-values increase, the y-values also increase.
- If we make the x-values smaller and smaller, the y-values will get smaller, but will not reach $y = 0$, telling us that $\boldsymbol{y = 0}$ is an **asymptote** of the graph.
- The $\boldsymbol{y}$**-intercept** is at $\boldsymbol{y = 1}$, because $2^0 = 1$

Solution:

x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



Example 2:

Consider the function $f(x) = \left(\frac{1}{2}\right)^x$. Complete the table below and plot the points on the Cartesian plane.

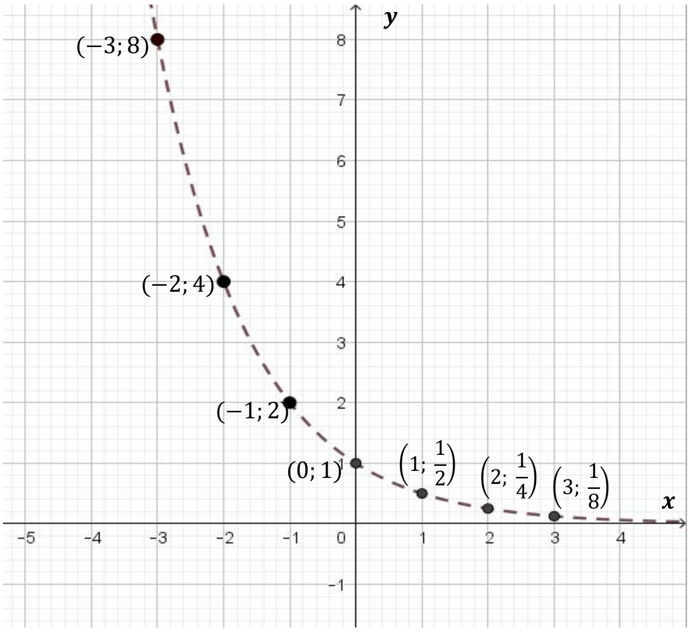
x	-3	-2	-1	0	1	2	3
$f(x)$							

What do we see:

- If we join the points, we see a graph that is **decreasing**: as the x-values increase, the y-values decrease.
- If we make the x-values bigger and bigger, the y-values will get smaller, but will not reach $y = 0$, telling us that $\boldsymbol{y = 0}$ is an **asymptote** of the graph.
- The ***y*-intercept** is at $\boldsymbol{y = 1}$, because $\left(\frac{1}{2}\right)^0 = 1$

Solution:

x	-3	-2	-1	0	1	2	3
$f(x)$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



SUMMARY:

In the graph of $f(x) = b^x$:

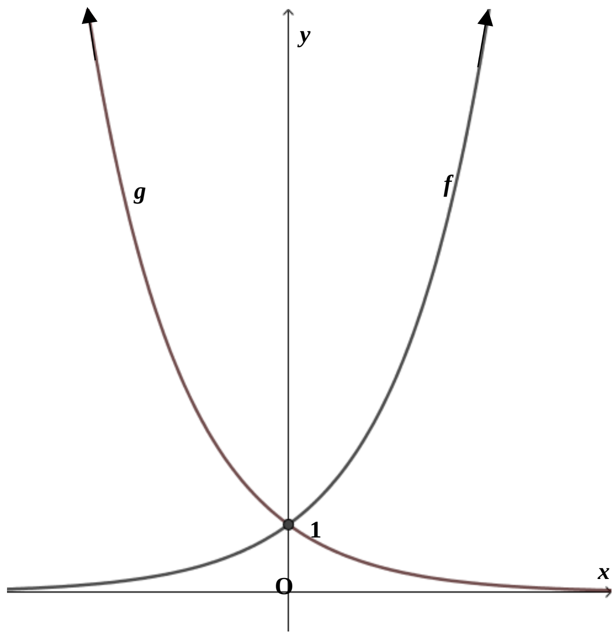
- If $\boldsymbol{b > 1}$, the graph is **increasing**
- If $\boldsymbol{0 < b < 1}$, (a fraction) the graph is **decreasing**
- $\boldsymbol{y = 0}$ is a **horizontal asymptote** to the graph
- the ***y*−intercept** is $\boldsymbol{y = 1}$, since $b^0 = 1$
- there is **no *x* −intercept** since there's nowhere where $y = 0$

NOTE: Why limit *b* to $\boldsymbol{b > 1}$ or $\boldsymbol{0 < b < 1}$?

- If $b = 0$, then $y = 0^x = 0$, which is a straight line and not an exponential graph (Note 0^0 is undefined)
- If $b = 1$, then $y = 1^x = 1$, which is also a straight line and not an exponential graph
- If $b < 1$, (negative) for e.g. $y = (-2)^x$, then for some values of x, the y-values are non-real, e.g. $(-2)^{\frac{1}{2}} = \sqrt{-2}$ which is Non-ℝ (you'll learn about this in Gr 11)

In the diagram:

$f(x) = b^x, b > 1$ and $g(x) = b^x, 0 < b < 1$



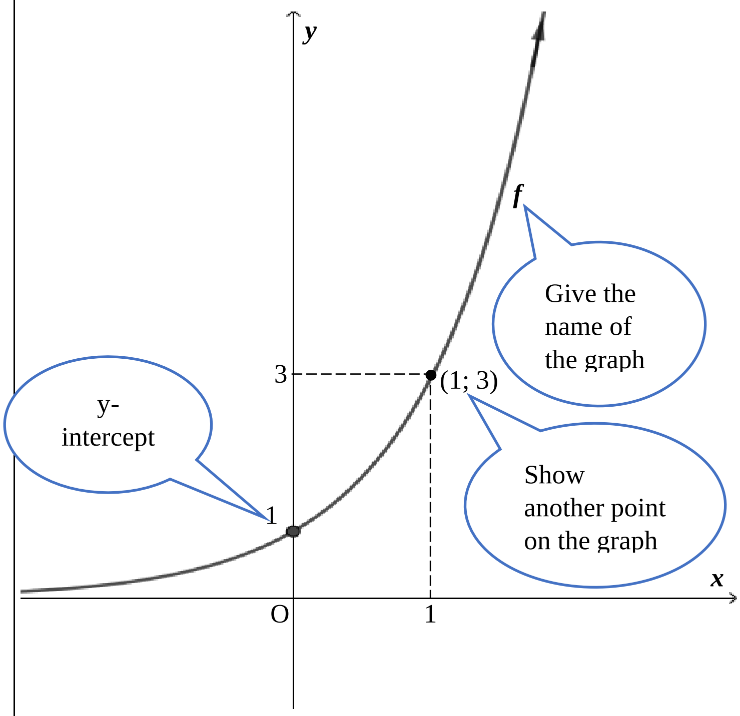
How to draw the exponential function (without using a table)

Example 3:

Draw the graph of $f(x) = 3^x$ on the Cartesian plane

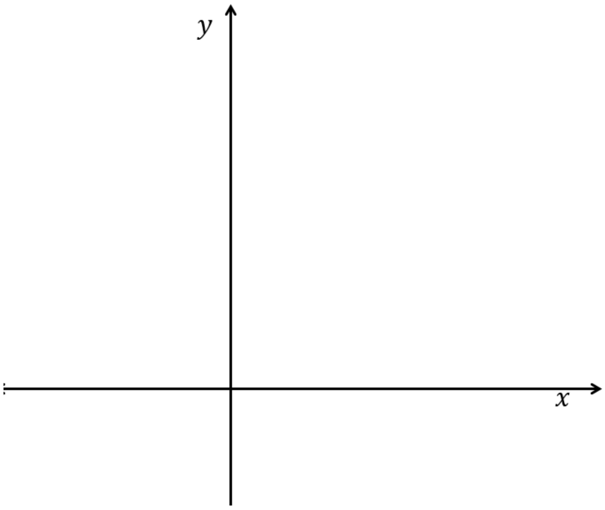
Solution:

- ***b* > 1**, *so, the graph is* ***increasing.***
- *Draw an increasing exponential graph*
- *$y-$intercept is at $y = 1$*
- *Choose any other point, say at x = 1 and calculate the corresponding y-value by substitution ⇒ y = 3*
- *Plot the point $(1;\ 3)$ on the graph*
- *Name the graph f*

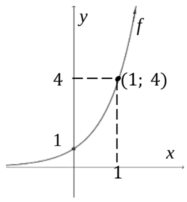


Can you?

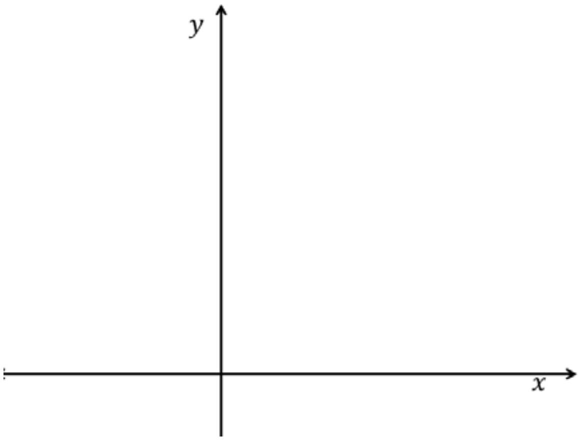
1. Draw the graph of $f(x) = 4^x$ on the Cartesian plane



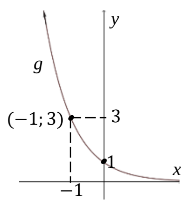
Solution:



2. Draw the graph of $g(x) = \left(\frac{1}{3}\right)^x$ on the Cartesian plane



Solution:



Lesson 1b	Sketch exponential function $y = ab^x + q$ using characteristics; **the influence of a** ($q = 0$)

Example 4:

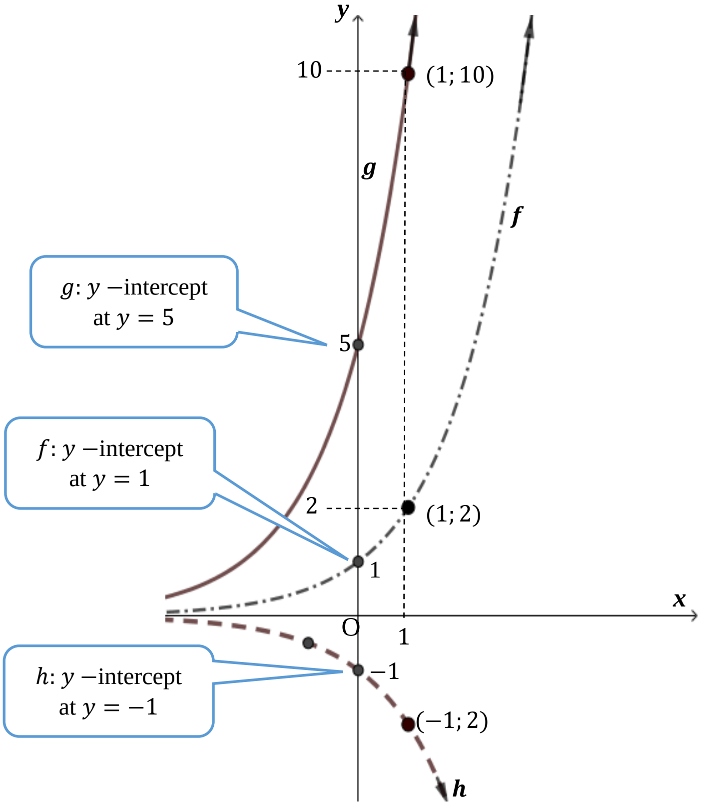
$a = 1$

Sketch the graph of $f(x) = 2^x$ ('mother function') and $g(x) = 5.2^x$ on the same set of axes. Also draw $h(x) = -2^x$

$a = 5$

$a = -1$

The graphs will look like this:



Solution:

If we use a table, we find the following:

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$g(x)$	$\frac{5}{4}$	$\frac{5}{2}$	5	10	20
$h(x)$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4

$-2^x \neq (-2)^x$
but
$-2^x = (-1).2^x$

All the y-values of f are multiplied by 5 to get $g(x)$

We see:

- the graphs have different **y −intercepts (at $y = a$)**
- the graph of g is **steeper** than f − the **bigger** the value of a, the **steeper** the graph – it is more stretched upwards
- all the graphs have **$y = 0$ ($q = 0$)** as a **horizontal asymptote**
- $f(x) = 2^x$ and $h(x) = -2^x$ are **reflections** of each other in the **x −axis** (horizontal asymptote: $y = 0$)

Summary:

In the graph of $f(x) = a.b^x$

- the **y −intercept** is at **$y = a$**
- the **bigger** the value of a, the **steeper** (narrower) the graph
- the graph has **$y = 0$ ($q = 0$)** as **horizontal asymptote**
- if **$a < 0$**, then we have a **reflection** in the **x −axis**
- the **domain** of f is: $x \in \mathbb{R}$, since the graph exists for **ALL** x −values
- the **range** of f is: $y > 0$, since the graph only exists for the y −values from the asymptote upwards (excluding the asymptote)

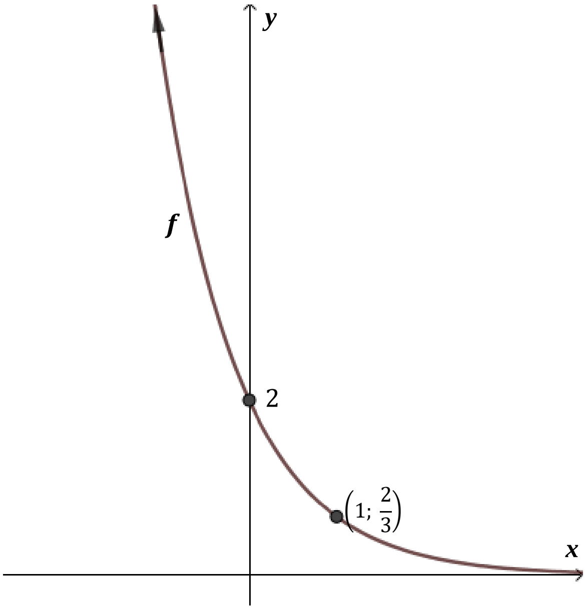
RECALL
Domain: all the **x −values** for which the graph exists

RECALL
Range: all the **y −values** for which the graph exists

Example 5:

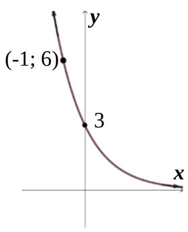
(a) Draw the graph of $f(x) = 2.\left(\frac{1}{3}\right)^x$
(b) Determine the domain and the range of f

Solution:
(a)
- $\mathbf{0 < b < 1}$, *so, the graph is* ***decreasing.***
- *Draw a decreasing exponential graph*
- *$y-$intercept is at $y = 2$*
- *Choose any* ***other point****, say at $x = 1$ and calculate the corresponding y-value by substitution $\Rightarrow y = \frac{2}{3}$*
- *Plot the point $(1; \frac{2}{3})$ on the graph*
- *Name the graph*

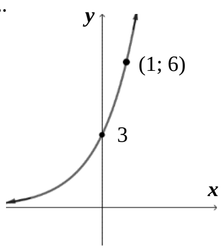


(b) Domain: $x \in \mathbb{R}$
Range: $y > 0$

Solutions:
1.(a)



2.



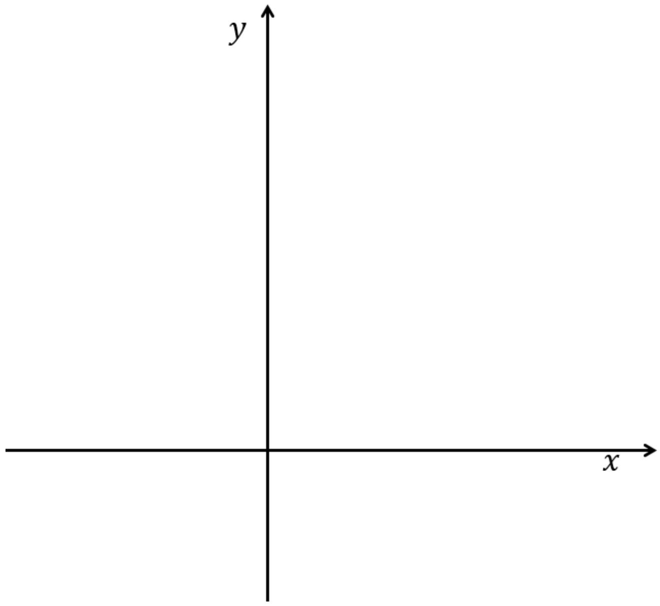
1(b) Domain: $x \in \mathbb{R}$
Range: $y > 0$

1(c) Decreasing.

Can you?

1. (a) Draw the graph of $f(x) = 3.\left(\frac{1}{2}\right)^x$
   (b) What is the domain and range of f
   (c) Is the graph of f increasing or decreasing?

Solution:
(a)

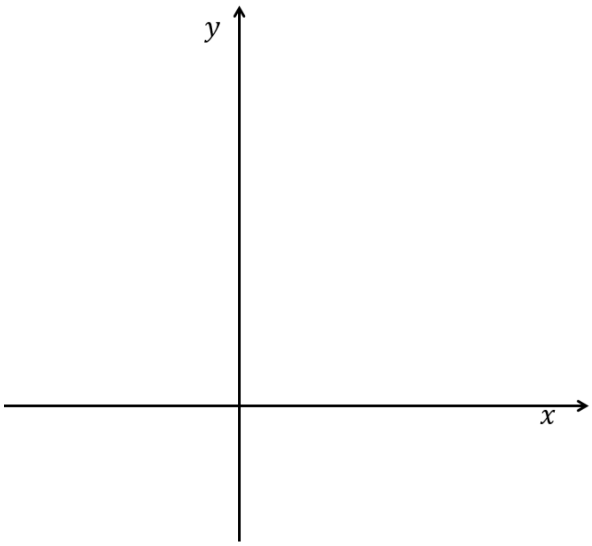


(b) domain:
range:

(c)

2. Draw the graph of $g(x) = 3.2^x$

Solution:

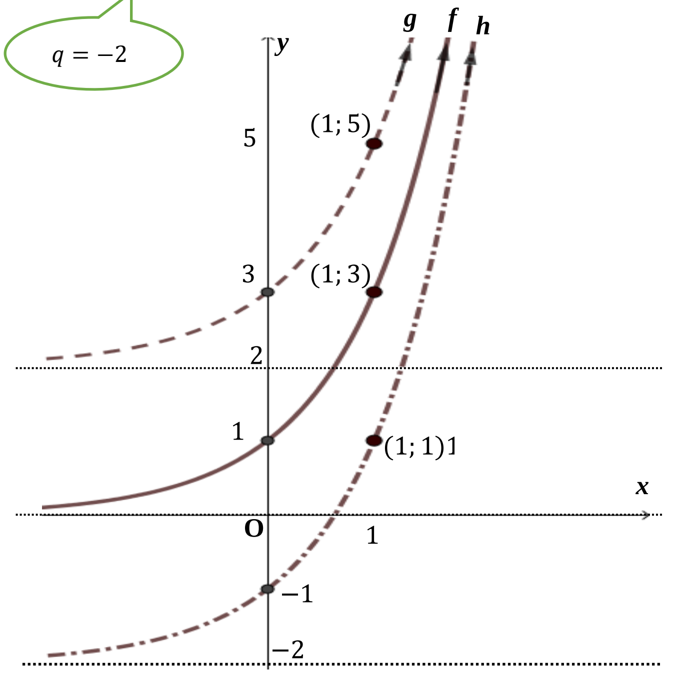


Lesson 1b	Sketch exponential function $y = ab^x + q$ using characteristics; **the influence of q**

Example 6:

$q = 0$ $q = 2$

Sketch the graph of $f(x) = 3^x$, $g(x) = 3^x + 2$ and $h(x) = 3^x - 2$ on the same set of axes



Solution:

If we use a table, we find the following:

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	$2\frac{1}{9}$	$2\frac{1}{3}$	3	5	11
$h(x)$	$-1\frac{8}{9}$	$-1\frac{2}{3}$	-1	1	7

$3^{-1} - 2 = -\frac{5}{3} = -1\frac{2}{3}$

What do we see:

- All the graphs are increasing, since $b > 1$
- f has a horizontal asymptote at $y = 0$ $(q = 0)$
- g has a horizontal asymptote at $y = 2$ $(q = 2)$
- h has a horizontal asymptote at $y = -2$ $(q = -2)$
- the 'mother function' f has a y −intercept at $y = 1$
- the y −intercept of g is at $y = 3$; the 'mother function' has been shifted 2 units upwards $(q = 2)$
- the y −intercept of h is at $y = -1$; the 'mother function' has been shifted 2 units downwards $(q = -2)$
- the domain of ALL the graphs is $x \in \mathbb{R}$
- the range of f is $y > 0$ (asymptote at $y = 0$);
- the range of g is $y > 2$ (asymptote at $y = 2$)
- and the range of h is $y > -2$ (asymptote at $y = -2$)

In general:

If $f(x) = a.b^x + q$ where $b > 1$ or $0 < b < 1$, then:

- f is **increasing** if $\boldsymbol{b > 1}$ and **decreasing** if $\boldsymbol{0 < b < 1}$
- f has a horizontal **asymptote** at $\boldsymbol{y = q}$
- The $\boldsymbol{y}$ **−intercept** is at $\boldsymbol{y = a + q}$ OR determine y by letting $x = 0$
- The $\boldsymbol{x}$ **−intercept** (if any) can be determined by solving $\boldsymbol{y = 0}$
- The graph of $y = a.b^x + q$ can be derived from the graph of the **'mother function'** $y = b^x$ by applying some transformations on it.
- The **domain** is $\boldsymbol{x} \in \mathbb{R}$
- The **range** is $\boldsymbol{y > q}$

Example 7:

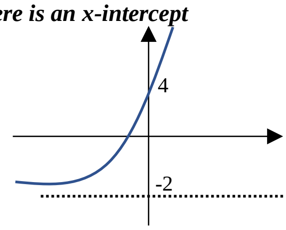
$a = 6; b = 3; q = -2$

Given $f(x) = 6.3^x - 2$

(a) Determine the intercepts of f with the axes.
(b) Give the equation of the horizontal asymptote.
(c) Sketch the graph of f
(d) Give the domain and range of f

Solution:

*We can make a rough sketch of f by just looking at the y-intercept and asymptote**: since b>1 the graph will be increasing; asymptote will be at $y = -2$ and y-int at $y = 4$ (6 − 2)$\Rightarrow$ there is an x-intercept***



(a) $\boldsymbol{y}$ **−intercept**: $\boldsymbol{y} = a + q = 6 - 2 = \mathbf{4}$

OR: let $x = 0 \quad \therefore y = 6.3^0 - 2 = 6.1 - 2 = 4$

$\boldsymbol{x}$ **−intercept**: Let $y = 0 \quad \therefore 0 = 6.3^x - 2$

$\therefore 2 = 6.3^x$

$\therefore \frac{2}{6} = 3^x = \frac{1}{3}$

$\therefore 3^x = 3^{-1}$

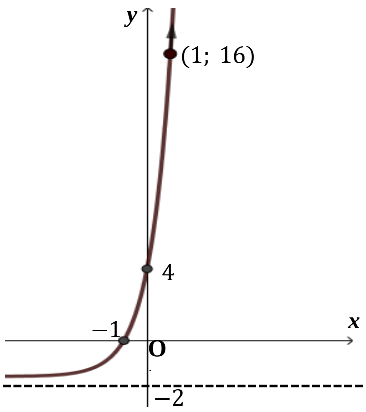
$\therefore \boldsymbol{x} = \mathbf{-1}$

(b) **Asymptote**: $y = q \quad \therefore \boldsymbol{y} = \mathbf{-2}$

(c) graph

- 1st draw in the asymptote;
- now draw an increasing exponential graph ($b > 1$) that goes through $y = 4$ and $x = -1$;
- choose any other point, say $x = 1$ and determine the corresponding y −value $\Rightarrow y = 6.3^1 - 2 = 16$
- Plot the point (1; 16) on the graph
- Name the graph

The graph should look like this:



(d) **domain**: $x \in \mathbb{R}$
range: $y > -2$

Can you?

1. Given $f(x) = 2.4^x + 2$

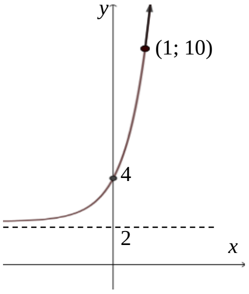
(a) Determine the intercepts of f with the axes.
(b) Give the equation of the horizontal asymptote.
(c) Sketch the graph of f
(d) Give the domain and range of f

Hint:
1st draw a rough sketch of $f \Rightarrow$ no x −int

2. Draw the graph of $g(x) = -3^x + 3$

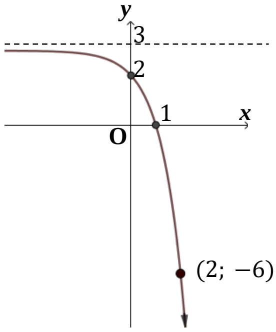
Solutions:

1.



(d) **domain**: $x \in \mathbb{R}$
range: $y > 2$

2.

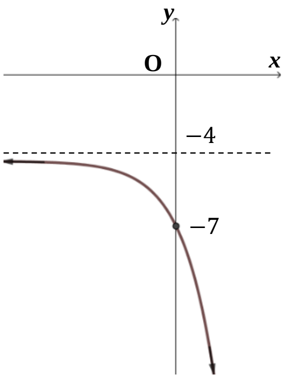


Lesson 1c	Finding the equation of exponential function given the graph

Example 8:

Find the equation of the given graph in the form:

$f(x) = a.2^x + q$



Solution:

$f(x) = a.2^x + q$

$q = -4$ (horizontal asymptote at $y = -4$)

$\Rightarrow f(x) = a.2^x - 4$

y-intercept

Substitute $(0; -7)$ in equation:

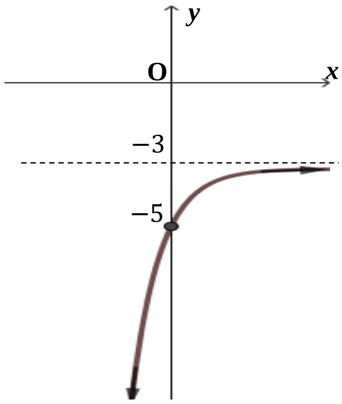
$\therefore -7 = a.2^0 - 4$
$\therefore -7 + 4 = a.1$
$\therefore a = -3$

$\Rightarrow f(x) = -3.2^x - 4$

Can you?

Find the equation of the given graph in the form:

$f(x) = a.\left(\frac{1}{5}\right)^x + q$



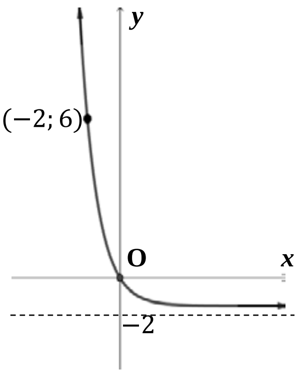
Solution:

Solution:

$f(x) = -2.\left(\frac{1}{5}\right)^x - 3$

Example 9:

Find the equation of the given graph in the form:
$g(x) = a.b^x + q$



Solution:

$g(x) = a.b^x + q$

horizontal asymptote is at $y = -2$
$\therefore g(x) = a.b^x - 2$

graph goes through origin $(0;\ 0)$
Subst. $(0;\ 0)$ in equation: $\Rightarrow 0 = a.b^0 - 2$
$\therefore 2 = \text{a}.1 \Rightarrow \boldsymbol{a = 2}$
$\therefore g(x) = 2.b^x - 2$

since graph is decreasing, b is a fraction

Subst. point $(-2;\ 6)$ in equation:
$\therefore 6 = 2.b^{-2} - 2 \quad \Rightarrow 6 + 2 = 2.b^{-2} \quad \therefore \frac{8}{2} = 4 = b^{-2}$

$\therefore \quad (b^{-2})^{-\frac{1}{2}} = 4^{-\frac{1}{2}}$

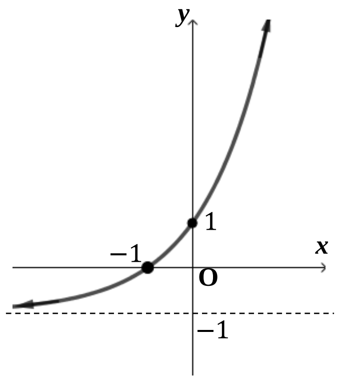
$\therefore \quad b = (2^2)^{-\frac{1}{2}} = 2^{-1}$

$\therefore \quad \boldsymbol{b = \frac{1}{2}}$

$\therefore$equation: $g(x) = 2.\left(\frac{1}{2}\right)^x - 2$

Can you?

Find the equation of the given graph in the form:
$g(x) = a.b^x + q$



Solution:

Solution:

$g(x) = 2.2^x - 1$

Conclusion: If $f(x) = a.b^x + q$ where $b > 1$ or $0 < b < 1$, then:

- o f is **increasing** if $\boldsymbol{b > 1}$ and **decreasing** if $\boldsymbol{0 < b < 1}$
- o f has a horizontal **asymptote** at $\boldsymbol{y = q}$
- o The $\boldsymbol{y}$ **−intercept** is at $\boldsymbol{y = a + q}$ OR determine y by letting $\boldsymbol{x = 0}$
- o The $\boldsymbol{x}$ **−intercept** (if any) can be determined by solving $\boldsymbol{y = 0}$
- o The graph of $y = a.b^x + q$ can be derived from the graph of the **'mother function'** $y = b^x$ by applying some transformations on it.
- o The **domain** is $\boldsymbol{x} \in \mathbb{R}$
- o The **range** is $\boldsymbol{y > q}$

- Determine the equation of a given graph by using: $y = a.b^x + q$
  - o Get the value of q from the horizontal asymptote
  - o Find the value of a by substituting the other point into the equation
  - o Usually b will be given: if the graph is increasing then $b > 1$; when the graph is decreasing, b is a fraction otherwise substitute another point to determine b

EXERCISES on the Exponential function
Siyavula Chapter 5 pg. 145
pg. 157 Ex 5.5; pg. 185 End of chapter Ex nrs. 3; 10f; 11; 12

Mind Action Series Grade 10
pg. 131– 133 Ex 4 a – h; pg. 153 – 154 Ex 11; pg. 116 – 117 Consolidation and Revision

Classroom Mathematics pg's.142–178: