SUBJECT and GRADE	MATHEMATICS GR 11
TERM 2	Week 6
TOPIC	TRIGONOMETRY
AIMS OF LESSON	

- To determine the signs of the different Trigonometric ratios in the different quadrant using the CAST diagram.
- To show how to, to reduce a trigonometrice ratio of angles(which are not acute) i.e $(180^0 \pm x)$ and $(360^0 \pm x)$, where x is an acute angle to the trigonometric ratio of the acute angle x .
- To show how to, reduce a trigonometrice ratio of the angles $(90^0 \pm x)$ to the trigonometric ratio of x.
- To show how to write the Trigonometric ratio of a negative angle as the trigonometric ratio of a positive angle.
- To prove Identities.

RESOURCES	Paper based resources	Digital resources
	Please go to the section on Trigonometry in your textbook.	https://www.youtube.com/watch?v=rS1T_b4WZzA https://www.youtube.com/watch?v=Dt5oEd2QWis

INTRODUCTION:

Dear Learner you have been introduced to Trigonometry in grade 10. Work through the following to revise the grade 10 Trigonometry you will require for the rest of this lesson:

Revision

- Trigonometric ratios

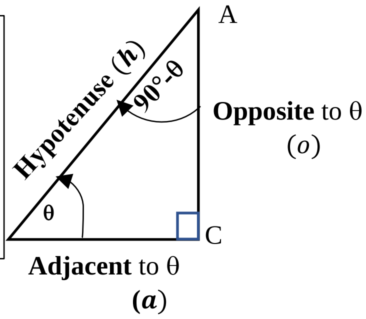
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{o}{h}\quad , \cos\theta = \frac{\text{adjacent}}{\text{hypotenues}} = \frac{a}{h}\quad , \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{o}{a}$$

Soh Cah Toa

OR

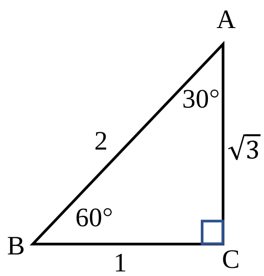
sinoh cosah tanoa

(make your OWN rhyme to remember the ratios as they are **VERY IMPORTANT**)



- Special Angles, 0°, 30°, 45° 60° and 90° you have been introduced to.

60°/ 30° triangle



$\sin 60° = \frac{o}{h} = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{a}{h} = \frac{1}{2}$

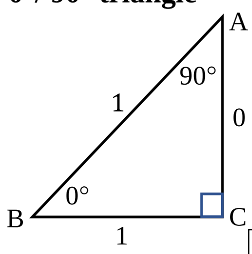
$\tan 60° = \frac{o}{a} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\sin 30° = \frac{o}{h} = \frac{1}{2}$

$\cos 30° = \frac{a}{h} = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{o}{a} = \frac{1}{\sqrt{3}}$

0°/ 90° triangle



$\sin 0° = \frac{o}{h} = \frac{0}{1} = 0$

$\cos 0° = \frac{a}{h} = \frac{1}{1} = 1$

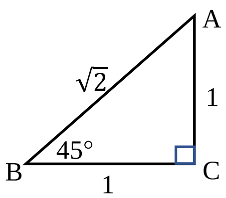
$\tan 0° = \frac{o}{a} = \frac{0}{1} = 0$

$\sin 90° = \frac{o}{h} = \frac{1}{1} = 1$

$\cos 90° = \frac{a}{h} = \frac{0}{1} = 0$

$\tan 90° = \frac{o}{a} = \frac{1}{0} = \text{undefined}$

45° triangle



$\sin 45° = \frac{o}{h} = \frac{1}{\sqrt{2}}$

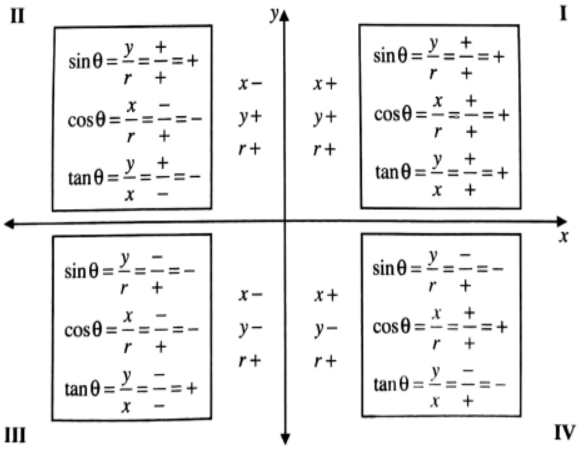
$\cos 45° = \frac{a}{h} = \frac{1}{\sqrt{2}}$

$\tan 45° = \frac{o}{a} = \frac{1}{1} = 1$

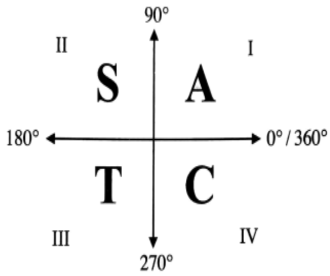
Revision

SIGNS OF TRIGONOMETRIC RATIOS

The following diagram shows the signs of x, y and r , as well as the signs of the three trigonometric ratios in each of the four quadrants.



We summarise this in the so-called CAST diagram



The letters A, S, T and C indicate which ratio(s) are **positive** in each quadrant.

In quadrant I: **A**ll trig ratios are positive

In quadrant II: **s**in is **positive** and all other ratios are negative

In quadrant III: **tan** is **positive** and all other ratios are negative

In quadrant IV: **c**os is **positive** and all other ratios are negative

Example 1:

If $3\tan\theta - 4 = 0$ and $\theta \in (180°; 360)$

Determine the value of

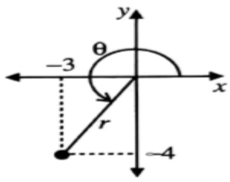
$25\sin^2\theta - 5\cos\theta$

with the aid of a diagram and without the use of a calculator**.**

Solution:

$3\tan\theta = 4 \qquad \therefore \tan\theta = +\frac{4}{3}$

$\tan\theta = \frac{4}{3} = \frac{y}{x}$



$x^2 + y^2 = r^2$

$(-3)^2 + (-4)^2 = r^2$

$25 = r^2$

$r = 5$

$\therefore\ 25\sin^2\theta - 5\cos\theta$

$= 25\left(\frac{-4}{5}\right)^2 - 5\left(\frac{-3}{5}\right)$

$= 16 + 3 = 19$

Steps:

1) Isolate the trig ratio and the sign
2) **Identify the quadrant** – (tanθ is positive in 1st and 3rd quadrant), $\theta \in (180°; 360°)$- true for 3rd and 4th quadrant **(hence 3rd quadrant)**
3) Draw the right angled triangle in the 3rd quadrant on the cartesian plane.
4) In the ratio, $\tan\theta = +\frac{4}{3}$, two sides are given (i.e x and y), and you need to determine the 3rd side, r.
5) Note, $\sin^2\theta = (\sin\theta)^2$
6) Now determine $\sin\theta$ and $\cos\theta$ and Substitute it into the expression
7) Calculate the value of the expression

CAN YOU : Solve the following questions.

a) If $-5\sin\theta = 4$ and $\theta \in (90°\,; 270°)$, determine with the use of a diagram and without the use of a calculator the value of:

1) $5\cos\theta - 3\tan\theta$
2) $\frac{4}{\sin\theta} - \frac{3}{\cos\theta}$

b) If $3\tan\beta = -2$ and $\beta \in (0°\,; 180°)$, determine with the use of a diagram and witout the use of a calculator the value of:

1) $\cos\beta$
2) $2\sin^2\beta - 1$
3) $\sqrt{13}\ \cos\beta - 13\sin^2\beta$

__Answers__:

a. 1) -7

a. 2) 0

b. 1) $\left(-\frac{3}{\sqrt{13}}\right)$

b. 2) $\left(-\frac{5}{13}\right)$

b. 3) -7

CONCEPTS AND SKILLS:

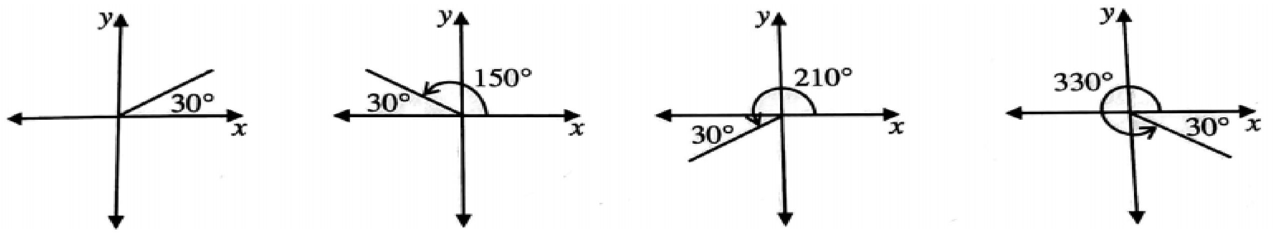
LESSON 1a: REDUCTION

Reduction is the process by which a trigonometric ratio of a **non-acute angle** is rewritten in terms of a ratio of an **acute angle.**

If we calculate the trigonometric ratios for the angles $30°;\ 150°;\ \ 210°;\ \ 330°$, using a calculator you will get:

$\sin 30° = \frac{1}{2}$	$\cos 30° = \frac{\sqrt{3}}{2}$	$\tan 30° = \frac{\sqrt{3}}{3}$
$\sin 150° = \frac{1}{2}$	$\cos 150° = -\frac{\sqrt{3}}{2}$	$\tan 150° = -\frac{\sqrt{3}}{3}$
$\sin 210° = -\frac{1}{2}$	$\cos 210° = -\frac{\sqrt{3}}{2}$	$\tan 210° = \frac{\sqrt{3}}{3}$
$\sin 330° = -\frac{1}{2}$	$\cos 330° = \frac{\sqrt{3}}{2}$	$\tan 330° = -\frac{\sqrt{3}}{3}$

Notice that each of the trigonometric ratios in the same column have the same numerical value for all four of these angles (although the signs differ). You will also note that the terminal arm of each of these angles make an acute angle of 30° with the x-axis.



Each of the trigonometric ratios of each of these angles can be written in terms of a trigonometric ratio of $30°$ as follows:

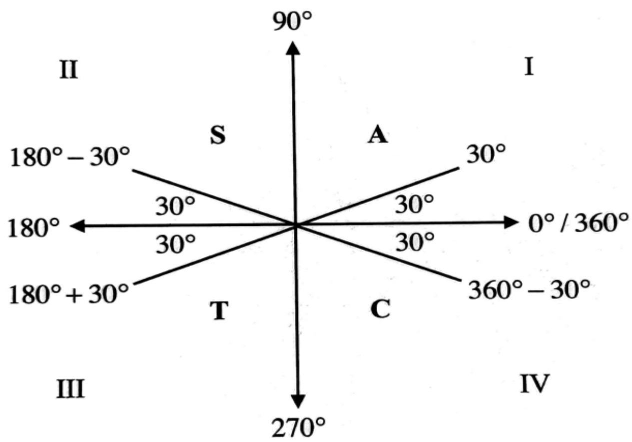
- The angles $150°,\ 210°,$ and $330°$ are rewritten in terms of $30°$ and either $180°$ or $360°$:
  $150° = 180° - 30°$ $\qquad$ $210° = 180° + 30°$ $\qquad$ $330° = 360° - 30°$

In the given cartesian plane notice that,

$180° - 30°$: is an angle in the 2nd quadrant, in fact any angle written as $(180° - x)$ where x is an acute angle is in the 2nd quadrants,

$180° + 30°$: is an angle in the 3rd quadrant, in fact any angle written as $(180° + x)$ where x is an acute angle is in the 3rd quadrant,

$360° - 30°$: is an angle in the 4th quadrant, in fact any angle written as $(360° - x)$ where x is an acute angle is in the 4th quadrant,



- The sign is determined by the quadrant, according to the CAST diagram, and the $180° -$, $180° +$, or $360° -$ is dropped

In the **second quadrant** $(180° - \cdots)$, **sin is positive** and cos and tan are negative:
$\sin 150° = \sin(180° - 30°) = +\sin 30°$
$\cos(180° - 30°) = -\cos 30°$
$\tan 150° = \tan(180° - 30°) = -\tan 30°$

In the **third quadrant** $(\mathbf{180°} + \cdots)$, **tan is positive** and cos and sin are negative:
$\sin(180° + 30°) = -\sin 30°$
$\cos 210° = \cos(180° + 30°) = -\cos 30°$
$\tan 210° = \tan(180° + 30°) = +\tan 30°$

In the **fourth quadrant** $(\mathbf{360°} - \cdots)$, **cos is positive** and sin and tan are negative:
$\sin 330° = \sin(360° - 30°) = -\sin 30°$
$\cos 330° = \cos(360° - 30°) = +\cos 30°$
$\tan 330° = \tan(360° - 30°) = -\tan 30°$

Generalising Reduction Formulae for $(180° \pm \theta)$ & $(360° \pm \theta)$

Consider a circle with centre O$(0;\ 0)$ and radius r. Suppose that $P(a;b)$, $P'(-a;b)$, $P''(-a;-b)$ and $P'''(a;-b)$ are all points on the circle. In the sketches below an acute angle θ is considered.

OP is a ray with point $P(a;b)$, and OP has been rotated through an angle θ from the x- axes in an anti-clockwise direction. That is the sketch with P in the 1st quadrant. All the trigonometric ratios can then be written down for θ, as has been done below.

OP' is a ray with point $P'(-a;b)$, this will be the case when the angle between OP' and the negative x-axis is θ. That means the angle through which OP' is rotated to get to $P'(-a;b)$ must be $(180° - \theta)$. All the trigonometric ratios for $(180° - \theta)$ can also be wrtten down as has been done below.

OP'' is a ray with point $P''(-a;\ -b)$, this will be the case when the angle between OP'' and the negative x-axis is θ. That means the angle through which OP'' is rotated to get to $P''(-a;-b)$ must be $(180° + \theta)$. All the trigonometric ratios for $(180 + \theta)$ can also be written down as has been done below.

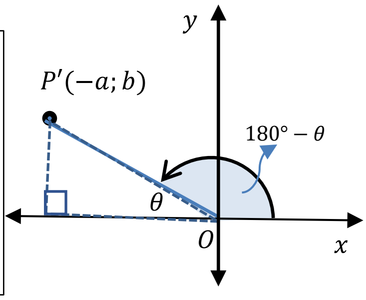
OP''' is a ray with point $P'''(a;\ -b)$, this will be the case when the angle between OP''' and the x-axis is θ. That means the angle through which OP''' is rotated to get to $P'''(a;-b)$ must be $(360° - \theta)$. All the trigonometric ratios for $(360° - \theta)$ can also be wrtten down as has been done below.

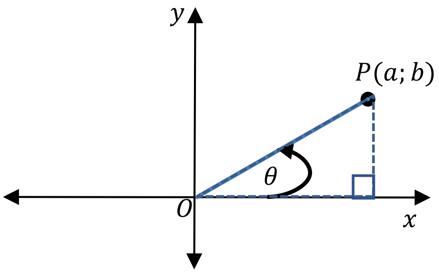
$(180° - \theta)$

$\sin(180° - \theta) = \frac{b}{r}$

$\cos(180° - \theta) = \frac{-a}{r} = -\frac{a}{r}$

$\tan(180° - \theta) = \frac{b}{-a} = -\frac{b}{a}$



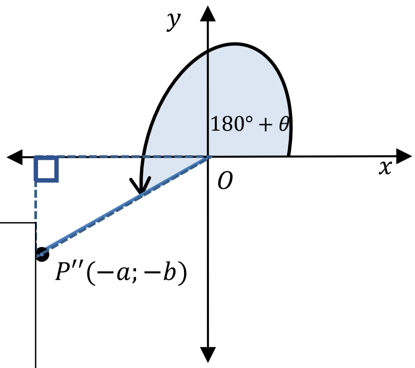


θ

$\sin\theta = \frac{b}{r}$

$\cos\theta = \frac{a}{r}$

$\tan\theta = \frac{b}{a}$

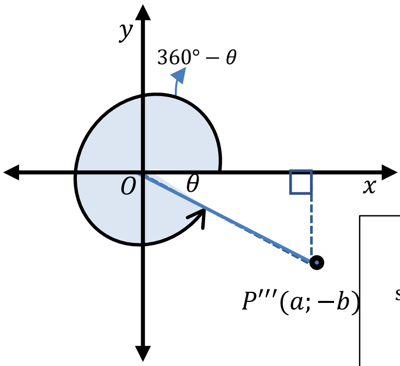


$(180° + \theta)$

$\sin(180° + \theta) = \frac{-b}{r} = -\frac{b}{r}$

$\cos(180° + \theta) = \frac{-a}{r} = -\frac{a}{r}$

$\tan(180° + \theta) = \frac{-b}{-a} = \frac{b}{a}$



$(360° - \theta)$

$\sin(360° - \theta) = \frac{-b}{r} = -\frac{b}{r}$

$\cos(360° - \theta) = \frac{a}{r} = \frac{a}{r}$

$\tan(360° - \theta) = \frac{-b}{a} = -\frac{b}{a}$

Note the following conclusions that can be made from the above:

Quadrant I **A**	Quadrant II **S**	Quadrant III **T**	Quadrant IV **C**
$\sin\theta = +\sin\theta$	$\text{Sin}(180° - \theta) = +\sin\theta$	$\text{Sin}(180° + \theta) = -\sin\theta$	$\text{Sin}(360° - \theta) = -\sin\theta$
$\cos\theta = +\cos\theta$	$\cos(180° - \theta) = -\cos\theta$	$\cos(180° + \theta) = -\cos\theta$	$\text{Cos}(360° - \theta) = +\cos\theta$
$\tan\theta = +\tan\theta$	$\tan(180° - \theta) = -\tan\theta$	$\tan(180° + \theta) = +\tan\theta$	$\text{Tan}(360° - \theta) = -\tan\theta$

The above will always be true. You will have to remember this. Notice that for the angles, $(180° - \theta)$, $(180° + \theta)$ and $(360° - \theta)$ listed above, the angle is replaced with the acute angle θ. The sign changes, to determine this sign the CAST diagram is used. Make a rhymn for you to remember the CAST diagram as it will enable you to determine in which quadrant each of the different trigonometric ratios are positive, and then by default negative.

Example	Solution	Steps
Example 2: Simplify the following without the use of a calculator: $\text{Sin}^2(180° + \theta)$	**Solution:** $\text{Sin}^2(180° + \theta)$ $= [\sin(180° + \theta)]^2$ $= [-\sin\theta]^2$ $= \sin^2\theta$	**Steps:** 1) Note, $\sin^2\theta = (\sin\theta)^2$ 2) Determine the **quadrant** of the given angle, $(180° + \theta)$. It is 3rd quadrant, in 3rd quadrant from CAST diagram, sine is negative. Write down the negative sign. 3) Write down $\boldsymbol{\sin\theta}$
Example 3: Reduce the following to a trigonometric ratio of an acute angle a) $\sin 150°$ b) $\tan 320°$	**Solution:** **a)** $\sin 150°$ $= \sin(180° - \mathbf{30°})$ $= +\sin \mathbf{30°}$ b) $\tan 320°$ $= \tan(360° - \mathbf{40°})$ $= -\tan 40°$	**Steps:** 1) Determine the **quadrant** of the given angle, 150°. It is in the 2nd quadrant. 2) Rewrite the non-acute angle, 150° as $(180° - 30°)$. 3) Use the quadrant of the angle and the CAST diagram to determine the **sign** of sin, in the 2nd quadrant sin is positive. Write the sign down. 3) Write down the sin **30°.** **Repeat the same steps for b).**
Example 4 : Simplify $\dfrac{\cos(180° + x).\sin(360° - x)}{\cos(x - 180°).\sin(x + 360°)}$	**Solution:** $\dfrac{\cos(180° + x)\ .\sin(360° - x)}{\cos(180° - x).\sin(180° - x)}$ $= \dfrac{(-\cos x)\ (-\sin x)}{(-\cos x)\ (\sin x)}$ $= -1$	**Steps:** 1) Note,$(180° + x)$ is in 3rd quadrant, cos is negative in 3rd quadrant from CAST diagram, therefore, $\cos(180° + x) = -\cos x$. 2) As in step 1, you do the same for, $\sin(360° - x)$, $\cos(180° - x)$ and $\sin(180° - x)$ 3) Then Simplify, cancelling like factors.

CAN YOU:

1. Simplify without the use of a calculator: a) $\sin(180° - \theta) + \sin(360° - \theta)$ b) $\dfrac{\cos(360°-\theta).\tan(180°-\theta)}{\cos(180°-\theta)}$	**2.** Reduce the following to a trigonometric ratio of an acute angle a) $\cos 315°$ b) $\tan 120°$	**Answers:** 1a) 0 b) $\tan\theta$ 2 a) $\cos 45°$ b) $-\tan 60°$

Anlgles greater than 360° : $(\mathbf{360°} + \boldsymbol{\theta})$

What about $(360° + \theta)$? If you start at angle θ and you add 360°, it is equivalent to you rotating through 360°, that means you have completed a revolution and is thus back at θ, where you started. If we add or substract a full revolution (360° or any multiple of 360°) to any angle, we end up at the same position on the Cartesian plane as the original angle.

Another way to think of it is, since 0° and 360° are in the same position, $(\boldsymbol{\theta} + 360°)$ and $(\boldsymbol{\theta} + \mathbf{0}°)$ must be the same position, which is the position of the angle θ**.**

Please note that: $\theta + 360° = 360° + \theta$

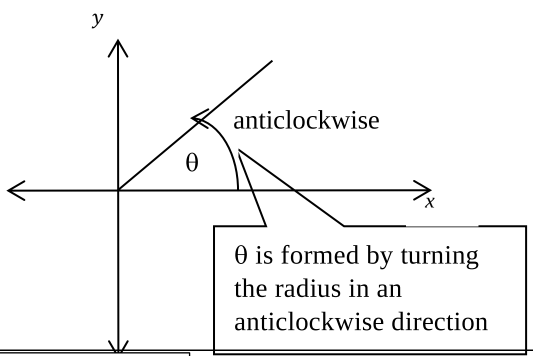
We can thus conclude the following for any angle θ.

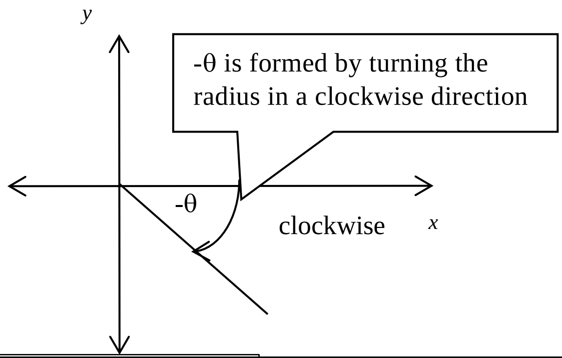
$\sin(360° + \theta) = \sin\theta$	$\cos(360° + \theta) = \cos\theta$	$\tan(360° + \theta) = \tan\theta$

Example	Solution	Steps
Examples 5: a)Determine the value of $\cos 495°$, without a calculator.	**Solution:** **a)** $\cos 495° = \cos(360° + 135°)$ $= \cos 135°$ $= \cos(180 - 45°)$ $= (-\cos 45°) = -\frac{1}{\sqrt{2}}$ This is obtained from special angles.	**Steps:** 1. The angle 495°, is greater than 360°, it can thus be wrtten as, $(360° + 135°)$. 2. Apply, $\cos(360° + \theta) = \cos\theta$ 3. To determine the value of, $\cos 135°$, express 135° in terms of an acute angle. $\cos 135°$, is an angle in 2nd quadrant and can be written as $(180° - 45°)$. 4. Apply, $\cos(180° - \theta) = \cos\theta$

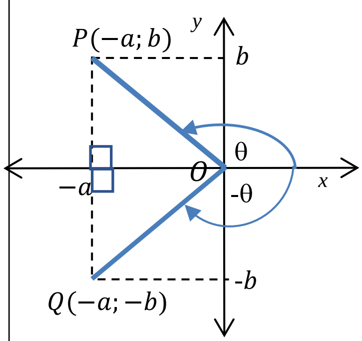
b) Simplify: $\tan(\theta - 180°)$	b) $\tan(\theta - 180°)$ $= \tan(\theta - 180° + 360°)$ $= \tan(180° + \theta)$ $= \tan\theta$	Steps: 1. Immediately it might not be obvious in what quadrant $(\theta - 180°)$ is. Adding 360°, does not change the position. It however helps us to better identify that $(\theta - 180°)$ is an angle in the 3rd quadrant. 2.Thus we able to apply the CAST diagram to, $\tan(180° + \theta)$
Example 6 : Simplify $\dfrac{\cos(360° + x).\sin(360° - x)}{\cos(x - 180°).\sin(x + 360°)}$	**Solution:** $\dfrac{\cos(360° + x).\sin(360° - x)}{\cos(x + 180°).\sin(x + 360°)}$ $= \dfrac{(\cos x)\,(-\sin x)}{(-\cos x)\,(\sin x)} = -1$	**Steps:** 1) Notice the angles $\sin(x + 360°)$, $\cos(360° + x)$ and $\cos(x - 180°)$ 2) Apply reduction formulae 3) Simplify
CAN YOU?		
a)Determine the value of $\tan 570°$, without a calculator.	b) Simplify: $\dfrac{\tan(360° - x).\sin(360° + x)}{\sin(x + 180°).\tan(x - 360°)}$	**Answers:** **a)** $\frac{\sqrt{3}}{3}$ **b)** **1**

LESSON 1b: NEGATIVE ANGLES

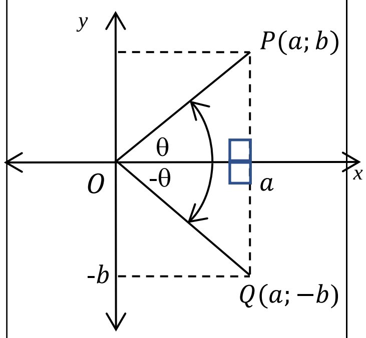




$P(-a; b)$ and $Q(-a; -b)$ are points on a circle of radius(r) and centre $O(0;\ 0)$.
OP has been rotated through an obtuse angle θ, in an anticlockwise direction.
OQas been rotated through an obtuse angle θ, in a clockwise direction.



$P(a; b)$ and $Q(a; -b)$ are two points on circle of radius(r) and centre $O(0;\ 0)$. OP, has been rotated through an acute angle θ, in an anticlockwise direction.
OQ has been rotated through an acute angle θ, in a clockwise direction.



θ (obtuse angle)	$-\theta$ (obtuse angle)
$\sin\theta = + \frac{b}{r}$	$\sin(-\theta) = \frac{-b}{r} = - \frac{b}{r}$
$\cos\theta = \frac{-a}{r} = - \frac{a}{r}$	$\cos(-\theta) = \frac{-a}{r} = - \frac{a}{r}$
$\tan\theta = \frac{b}{-a} = - \frac{b}{a}$	$\tan(-\theta) = \frac{-b}{-a} = + \frac{b}{a}$

θ (acute angle)	$-\theta$ (acute angle)
$\sin\theta = + \frac{b}{r}$	$\sin(-\theta) = \frac{-b}{r} = - \frac{b}{r}$
$\cos\theta = + \frac{a}{r}$	$\cos(-\theta) = + \frac{a}{r}$
$\tan\theta = + \frac{b}{a}$	$\tan(-\theta) = \frac{-b}{a} = - \frac{b}{a}$

Conclusion from the above tables:

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = +\cos\theta$
$\tan(-\theta) = -\tan\theta$

This is evident from the above sketches for acute angle θ as well as for obtuse angle θ.

Similarly it can be shown that this conclusion is true for any angle θ.

Example 7 : Simplify

$a)$ $\tan(\theta - 180°)$

b)

$$\frac{\sin(-\theta).\tan(\theta-180°).\cos(180°-\theta)}{\sin(180°-\theta).\tan(-\theta).\cos(-\theta)}$$

Solution:

a) $\tan(\theta - 180°) = \tan[-(180° - \theta)]$

$= -\tan(180° - \theta)$

$= -(-\tan\theta) = \tan\theta$

$b)$ $\dfrac{\sin(-\theta).\tan(\theta + 180°).\cos(180° - \theta)}{\sin(180° - \theta)\ .\tan(-\theta).\cos(-\theta)}$

$\dfrac{(-\sin\theta)\ (\tan\theta)(-\cos\theta)}{(\sin\theta)(-\tan\theta)\ (\cos\theta)}$

$= -1$

Steps:

a)

1. Note that, $(\theta - 180°)$ can be written as a negative angle, $[-(180° - \theta)]$
2. Then apply, $\tan(-\theta) = \tan\theta$
3. Then apply reduction formulae, $\tan(180° - \theta) = -\tan\theta$

Note: This question was done in Example 5b. Here it is done differently, and the same answer is obtained.

b)

1) Notice negative angles $\sin(-\theta)$, $\tan(-\theta)$ and $\cos(-\theta)$.
2) Also note the angle, $(\theta - 180°)$ to which, 360°, can be added.
3) Simplify

CAN YOU:

Simplify without the use of a calculator:

$$\frac{\tan(360° + \theta).\sin^2(-\theta)}{\sin(\theta - 180°)\ .\tan(-\theta)\sin(\theta - 360)}$$

Answer: 1

LESSON 1c: CO –FUNCTIONS

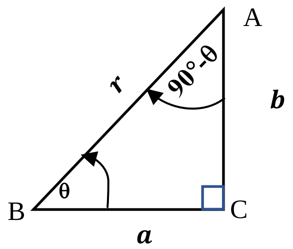
$90° - \theta$

Use your calculator and check if you agree with the following statements.

Sin $50° = \cos 40°$ and $\cos 50° = \sin 40°$
Sin $70° = \cos 20°$ and $\cos 70° = \sin 20°$

Note:
1. $50° + 40° = 90°$ and $70° + 20° = 90°$
2. sin changed to cos and cos changed to sin.

Lets investigate if the above will be true for any acute angle θ. Take any right angled triangle and let the one acute angle be θ, then the 3rd angle in the triangle is $(90° - \theta)$, because of angles in a triangle adding to 180°. Suppose a, b are two sides of the triangle and r is the hypotenuse. See the sketch.



We can write down the following from the sketch:

θ	$90° - \theta$
$\sin\theta = \dfrac{b}{r}$	$\sin(90° - \theta) = \dfrac{a}{r}$
$\cos\theta = \dfrac{a}{r}$	$\cos(90° - \theta) = \dfrac{b}{r}$

Conclusion for $(90° - \theta)$:

$\sin(90° - \theta) = \cos\theta$

$\cos(90° - \theta) = \sin\theta$

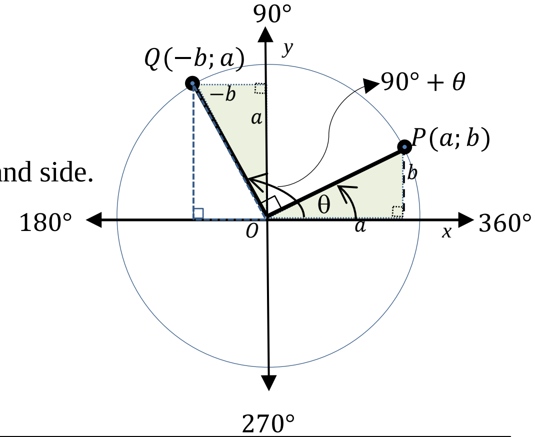
$90° + \theta$

$90° + \theta$

Consider a circle with radius r, and centre $O\ (0;\ 0)$ in a cartesian plane. Suppose that $P(a; b)$ is a point on the circle. Suppose that OP makes an angle θ with the positive x- axes. Q is the point obtained if OQ, is rotated through an angle $(90° + \theta)$ from the x-axes. Thus the angle between OQ and the y-axes must be θ. This is because the rotation from the x-axis to the y- axis is 90°. See the given sketch:

So when comparing the two green/ shaded triangles in the sketch, the following is true,

2. The hypotenuse is the same
3. θ is an angle in both
4. 90° is angle in both



Therefore these two green/shaded triangles are congruent, angle, angle and side.
The coordinates of Q, must be $Q(-b; a)$.
We can now determine the following:

θ	$90° + \theta$
$\sin\theta = \frac{y}{r} = \frac{b}{r}$	$\text{Sin}(90° + \theta) = \frac{y}{r} = \frac{a}{r}$
$\cos\theta = \frac{x}{r} = \frac{a}{r}$	$\text{Cos}(90° + \theta) = \frac{x}{r} = \frac{-b}{r}$

Conclusion for $(90° + \theta)$:

$$\sin(90° + \theta) = +\cos\theta$$

$$\cos(90° + \theta) = -\sin\theta$$

Summary for $(90 \pm \theta)$

$\text{Cos}(90° - \theta) = \sin\theta$ and $\sin(90° - \theta) = \cos\theta$

$\text{Cos}(90° + \theta) = -\sin\theta$ and $\sin(90° + \theta) = \cos\theta$

90° Rule

1) Determine the **quadrant** of the given angle
2) From the quadrant of the given angle determine the **sign** of the trigonometric ratio in the quadrant.
3) Write down the **co-function**
4) **Reduce** the angle in brackets to the **acute angle**.

Example 8:
Simplify without the use of calculator.

a)

$$\frac{\sin(90° - \theta).\ \sin\theta}{\cos\theta\ .\cos(90° + \theta)}$$

Solution:

a) $\frac{\sin(90° - \theta).\ \text{Sin}\,\theta}{\cos\theta\ .\cos(90° + \theta)}$

$\frac{(\cos\theta)\ (\sin\theta)}{(\cos\theta)\ (-\sin\theta)} = -1$

Steps: Substitute
1) $\sin(90° - \theta) = \cos\theta$
2) $\cos(90° + \theta) = -\sin\theta$
3) Simplify

b)

$$\frac{\cos(180° - x).\cos(90° - x).\tan(-x)}{\sin(180° + x)\ .\ \sin(90° + x)}$$

b)

$\frac{\cos(180° - x).\cos(90° - x).\tan(-x)}{\sin(180° + x)\ .\ \sin(90° + x)}$

$= \frac{(-\cos x).(\sin x)(-\tan x)}{(-\sin x)(\cos x)} = -\tan x$

1) $\cos(180° - x) = -\cos x$
2) $\cos(90° - x) = \sin x$
3) $\tan(-x) = -\tan x$
4) $\sin(180° + x) = -\sin x$
5) $\sin(90 + x) = \cos x$
6) Simplify

c)

$\sin 110° + \cos 200° + \sin^2 315°$

c) **Solution:**

$\sin 110° + \cos 200° + \sin^2 315°$

$= \sin(180° - 70°) + \cos(180° + 20°) + \sin^2(360° - 45°)$

$= \sin 70° - \cos 20° + \sin^2 45°$

$= \sin(90° - 20°) - \cos 20° + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \cos 20° - \cos 20° + \frac{1}{2} = \frac{1}{2}$

Steps:

1) Express each of the angles 110°; 200°; and 315° in terms of an acute angle.
2) $\sin 70° = \cos 20°$ co-function
3) Special angle $\sin 45° = \left(\frac{1}{\sqrt{2}}\right)$
4) simplify

CAN YOU:		
Simplify without the use of a calculator: **a)** $\frac{\cos(90°-\theta).\tan(180°-\theta)}{\cos(90°+\theta)}$	b) $\frac{\sin(90+\alpha)}{\sin(-\alpha)}$ c) $\frac{\sin 160°}{\cos 250°}$	**ANSWERS:** a) $\tan\theta$ b) -1 c) -1

LESSON 1d: IDENTITIES

Identity: When two expressions give the same result for all values of the variable (for which they both are defined)

Quotient Identity : $\boldsymbol{\tan\theta = \frac{\cos\theta}{\sin\theta}}$	Square Identity : $\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$ Also written as: $\boldsymbol{\sin^2\theta = 1-\cos^2\theta = (1-\cos\theta)(1+\cos\theta)}$ $\boldsymbol{\cos^2\theta = 1-\sin^2\theta = (1-\sin\theta)(1+\sin\theta)}$

Proving Identities -Steps

1. Work with the two sides separately
2. Write the ratios in terms of sin or cos hence write $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$.
3. Use the square identity where possible in the forms
   $\text{Sin}^2x + \cos^2x = 1$ or $\sin^2x = 1-\cos^2x$ or $\cos^2x = 1-\sin^2x$
   Note: $1-\cos^2x = (1-\cos x)(1+\cos x)and1-\sin^2x = (1-\sin x)(1+\sin x)$
4. Simplify and factorise as for an algebraic expression.
5. If there are fractions, find LCD and simplify.

Example 9: Prove the identity	**Solution:**	**Steps:**
a) $\frac{\text{Sin}^2x}{1-\cos x} = 1+\cos x$	LHS $= \frac{\sin^2x}{1-\cos x} = \frac{1-\cos^2x}{1-\cos x}$ $= \frac{(1-\cos x)(1+\cos x)}{1-\cos x}$ $= 1+\cos x$ = RHS	1) Rewrite $\sin^2x = 1-\cos^2x$ 2) Factorise : $1-\cos^2x = (1-\cos x)(1+\cos x)$ 3) Simplify
b) $(1+\tan^2x)(1-\sin^2x) = 1$	LHS $= (1+\tan^2x)(1-\sin^2x)$ $= \left(1+\frac{\sin^2x}{\cos^2x}\right)(1-\sin^2x)$ $= \left(\frac{\cos^2x+\sin^2x}{\cos^2x}\right)(\cos^2x)$ $= \frac{1}{\cos^2x} \times \cos^2x$ $= 1$ = RHS	**Steps:** 1) Rewrite $\tan^2x = \frac{\sin^2x}{\cos^2x}$ 2) Rewrite $1-\sin^2x = \cos^2x$ 3) Add fractions ,LCD 4) Rewrite $\sin^2x + \cos^2x = 1$ 5) Simplify
c) $\frac{1+\sin x}{\cos x} = \frac{\cos x}{1-\sin x}$	LHS $= \frac{1+\sin x}{\cos x}$ $= \frac{1+\sin x}{\cos x} \times \frac{1-\sin x}{1-\sin x}$ $= \frac{1-\sin^2x}{\cos x(1-\sin x)}$ $= \frac{\cos^2x}{\cos x(1-\sin x)}$ $= \frac{\cos x}{1-\sin x}$ = RHS	**Steps:** 1) If you can't use identities multiply LHS by 1 2) Find LCD and simplify 3) Replace: $1-\sin^2x = \cos^2x$ 4) Simplify

CAN YOU : Prove the following Identities

1. $\text{Sin}\, x\ .\cos x\ .\tan x = 1 - \cos^2 x$
2. $(\text{Sin}\, x + \cos x)^2 + (\sin x - \cos x)^2 = 2$
3. $\frac{\text{Cos}\, x}{1+\sin x} + \tan x = \frac{1}{\cos x}$
4. $\frac{1-\cos x}{\sin x} - \frac{\sin x}{1+\cos x} = 0$

ACTIVITIES:	
	Siyavula : (Exer 6-2 , Pg 252); (Exer 6-3, Pg 255); (Exer 6-4 , Pg 259); (Exer 6-6 , Pg 265) Mind Action series : (Exer 1- Pg 105); (Exer 2- Pg 108); (Exer 3- Pg 111); (Exer 4- Pg 114); (Exer 5- Pg 118); (Exer 6- Pg 123); (Exer 7- Pg 126) Platinum : (Exer 3- Pg 141); (Exer 5- Pg 144); (Exer 7 and 8- Pg 147); (Exer 9- Pg 149); (Exer 13- Pg 153) ; (Exer 14- Pg 154); Via Afrika : (Exer 2, Pg 154) ; (Exer 3, Pg 156) ; (Exer 5, Pg 159) ; (Exer 8- Pg 164) (Exer 13, Pg 169) ; (Exer 14- Pg 173)

CONSOLIDATION:

1. Memorize the identities:

   $\boldsymbol{\tan\theta = \frac{\cos\theta}{\sin\theta}}$ **&** : $\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$

2. Understand the CAST diagram.
3. In A and B below is a summary of the reduction formula: Application of the reduction formulae is vital. You can not memorize this table, so undersatnding how you can determine it is important.

   A: Reduction formulae for $(\boldsymbol{180° \pm \theta})$ **,** $(\boldsymbol{360° \pm \theta})$ **and** $(\boldsymbol{-\theta})$

$\sin(180° - \theta) = \sin\theta$	$\sin(180° + \theta) = -\sin\theta$	$\sin(360° - \theta) = -\sin\theta$	$\sin(360° + \theta) = \sin\theta$	$\sin(-\theta) = -\sin\theta$
$\cos(180° - \theta) = -\cos\theta$	$\cos(180° + \theta) = -\cos\theta$	$\cos(360° - \theta) = +\cos\theta$	$\cos(360° + \theta) = \cos\theta$	$\cos(-\theta) = +\cos\theta$
$\tan(180° - \theta) = -\tan\theta$	$\tan(180° + \theta) = +\tan\theta$	$\tan(360° - \theta) = -\tan\theta$	$\tan(360° + \theta) = \tan\theta$	$\tan(-\theta) = -\tan\theta$

B: Reduction formulae for $(\boldsymbol{90° \pm \theta})$

$$\sin(90° - \theta) = \cos\theta$$
$$\cos(90° - \theta) = \sin\theta$$
$$\sin(90° + \theta) = +\cos\theta$$
$$\cos(90° + \theta) = -\sin\theta$$

4. Know how to prove any identity.