SUBJECT and GRADE	MATHEMATICS GR 11	
TERM 2	Week 7	
TOPIC	TRIGONOMETRY	
AIMS OF LESSON	General solution to trigonometric equations	
RESOURCES	Paper based resources	Digital resources
	Textbook chapter about trigonometric equations.	https://www.youtube.com/watch?v=0jR_D84fecY https://www.youtube.com/watch?v=0jR_D84fecY
INTRODUCTION:	In this lesson the following pre-knowledge is essential: • CAST-diagram, reduction formules, trig identities, co-function & special angles. • Faktorising: Common factor & factorising of a quadratic expression. • Solving a quadratic equation. • $When\ is\ an\ expression\ or\ an\ equation\ undefined?$.	

Lesson 1: Trigonometric equations

Consider the equation $\sin\theta = \frac{1}{2}$.

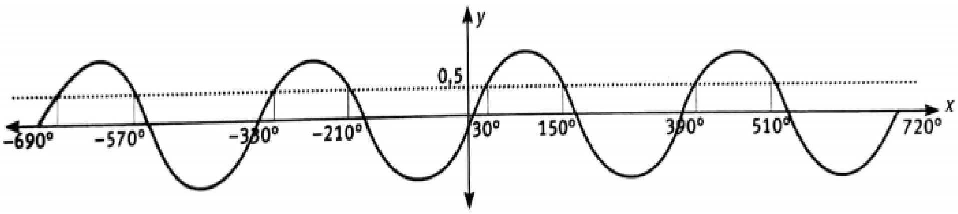
To solve this equation, we have to find all possible angles of θ for which $\sin\theta$ will equal $\frac{1}{2}$.

In the sketch: $y = \sin x$, is drawn over the interval, $[-720°; 720°]$. The line, $y = \frac{1}{2}$, has also been drawn.

So, the solution of the equation, $\sin\theta = \frac{1}{2}$, is in fact where the two graphs,

$y = \sin x$ and $y = \frac{1}{2}$

intersect.



You will notice from the graph that there are a number of **different angles that are all solutions** to this equation. $y = \sin x$ and $y = \frac{1}{2}$ continues indefinitely if the interval is not restricted, in much the same way the equation, $\sin\theta = \frac{1}{2}$, has **infinitely many solutions**

Note: The graph repeats itself every $360°$. So if you need to find the solution of, $\sin\theta = \frac{1}{2}$, for the interval, $[0°; 360°]$. All the other solutions will be obtained by adding or subtracting a multiple of $360°$.

- To determine these solutions, start by finding the **acute angle** from which all the solutions can be derived. This is called the **reference angle.** We find this angle by using the $\mathbf{sin^{-1}}$ button on the calculator.

  $\sin\theta = \frac{1}{2}$

  Reference $\angle = \sin^{-1}\left(\frac{1}{2}\right) = \mathbf{30°}$

- Next we look at the **sign** of the trigonometric ratio, to determine the correct quadrant

  $\sin\theta = +\frac{1}{2}$, so the trigonometric ratio is positive

  sin **is positive** in **quadrant I** and **quadrant II** (CAST-Diagram)

  I: $\theta = \text{reference angle}$ II : $\theta = 180° - \text{reference angle}$

  $\theta = 30°$ $\theta = 180° - 30° = 150°$

- Adding (or subtracting) **multiples of $360°$** to each of these will enable one to determine all possible solutions.

  I: $\theta = 30° + k.360°;\ k \in Z$ or **II:** $\theta = 150° + k.360°;\ k \in Z$

  This is refered as **The General Solution.**

For the **general solution** of a trigonometric equation of the form $\sin\theta = a$, $\cos\theta = a$ *or* $\tan\theta = a$

- Calculate the reference angle by applying $\sin^{-1}$; $\cos^{-1}$ *or* $\tan^{-1}$ to the numerical value of, "a" **ignoring the sign**
- Look at the sign of "a" and the trigonometric function to determine in which **two** quadrants the solution must be according the CAST-diagram

Quadrant I : $\boldsymbol{\theta = \text{ref angle} \quad + k.360°}$

Quadrant II : $\boldsymbol{\theta = 180° - \text{ref angle} \quad + k.360°}$

Quadrant III : $\boldsymbol{\theta = 180° + \text{ref angle} \quad + k.360°}$

Quadrant IV : $\boldsymbol{\theta = 360° - \text{ref angle} \quad + k.360°}$

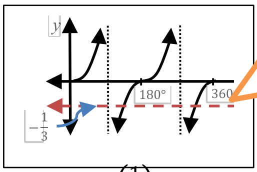
Example 1:

Determine the general solution of, $3\tan\theta + 1 = 0$.

Solution:

$3\tan\theta = -1$

$\tan\theta = -\left(\frac{1}{3}\right)$



NOTE: The tan graph has a period of 180°. That is the graph is repeated every 180°. So to get all solutions we add $k.180°$

$reference\angle = \tan^{-1}\left(\frac{1}{3}\right) = 18{,}43°$

II : $\theta = 180° - 18{,}43° \quad + k.180°$

$\therefore \quad \theta = 161{,}57° \quad + k.180°;\ k \in Z$

Answer:

$\theta = 161{,}57° + k.180° \quad k \in Z$

OR

NOTE: This will be the general solution if you go the route to finding the solution in both the 2nd and 4th quadrant.

$\theta = 161{,}57° + k.360°$ or $\theta = 341{,}57° + k.360°;\ k \in Z$

Steps

1) Isolate the trig ratio: $\left[\tan\theta = -\left(\frac{1}{3}\right)\right]$
2) Calculate the reference angle ignoring the sign of the trig ratio. ($\tan^{-1}$)
3) tan is negative in quadrants 2 and 4. In Quadrant II: $\theta = 180° - ref\angle$ Quadrant IV: $\theta = 360° - ref\angle$
4) As the tan graph repeats itself every 180°, we need not consider the solution in the 4th quadrant. This is because the solution in the 2nd quadrant plus 180°, is the solution in the 4th quadrant.
5) Thus for **$\tan\theta$, only one solution in the interval** $[0°; 180°]$ **is required then add $k.180°$ where $k \in Z$, for the general solution.** Simplify

Example 2: Determine the general solution of $\sin 2\theta = \frac{1}{4}$

Solution: $\sin 2\theta = \frac{1}{4}$

$reference\angle = \sin^{-1}\left(\frac{1}{4}\right) = 14{,}48°$

I : $2\theta = 14{,}48° \quad + k.360°$

$\theta = 7{,}24° \quad + k.180°;\quad k \in Z$

II : $2\theta = 180° - 14{,}48° \quad + k.360$

$2\theta = 165{,}52° \quad + k.360°$

$\theta = 82{,}76° \quad + k.180°;\quad k \in Z$

Answer:

$\theta = 7{,}24° \quad + k.180°$ or $\theta = 82{,}76° \quad + k.180°;\quad k \in Z$

Steps

1) Determine the reference angle as before ignoring the sign of the trig ratio.
2) sin is positive in quadrants 1 and 2. In, Quadrant I: $2\theta = ref\angle$ Quadrant II: $2\theta = 180° - ref\angle$
3) Add $k.360°$ to give the general solution.
4) To solve for θ divide by 2

Example 3:

Determine the general solution of $2\cos(\theta - 20°) = -\sqrt{3}$

Solution: $2\cos(\theta - 20°) = -\sqrt{3}$

$\cos(\theta - 20°) = -\frac{\sqrt{3}}{2}$

$reference\angle = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$

II : $\theta - 20° = 180° - 30° \quad + k.360°$

$\theta - 20° = 150° \quad + k.360°$

$\theta = 170° \quad + k.360°;\quad k \in Z$

III : $\theta - 20° = 180° + 30° \quad + k.360°$

$\theta - 20° = 210° \quad + k.360°$

$\theta = 230° + k.360;\quad k \in Z$

Answer:

Steps

1) Isolate the trig ratio, $\cos(\theta - 20°)$.
2) Determine the reference angle as before, ignoring the sign in front of $\sqrt{3}$.
3) cos is negative in quadrants 2 and 3. In, Quadrant II: $(\theta - 20°) = 180° - ref\angle$ Quadrant III: $(\theta - 20°) = 180° + ref\angle$
4) Add $k.360°$ to give the general solution.
5) To solve for θ add 20° on both sides of the equation.

$\theta = 170° + k.360°$ or $\theta = 230° + k.360°;\ k \in Z$

CAN YOU :

Determine the general solution of

1) $\cos x = 0{,}4$
2) $3 \sin x = 2$
3) $\tan(\alpha - 20°) = -3$
4) $3 \tan(3\theta - 30°) - 3 = 0$

Answers:

1. $x = 66{,}42° + k.360°\ or\ x = 293{,}58° + k.360°; k \in Z$
2. $x = 41{,}81° + k.360°\ or\ x = 138{,}19° + k.360°; k \in Z$
3. $\alpha = 128{,}43° + k.180°$

   OR

   $\alpha = 128{,}43° + k.360°\ or\ \alpha = 308{,}43° + k.360°; k \in Z$

4. $\theta = 25° + k.60°\ ; k \in Z$

   OR

   $\theta = 25° + k.120°\ of\ \theta = 85° + k.120°; k \in Z$

Lesson 2: Finding solutions in a given interval

Sometimes it will be required to find the solutions of a trigonometric equation within a specific interval.

Example 4: Solve for θ if

$\sin\theta = -\frac{1}{5}$ and $\theta \in [-360°;\ 360°]$.

Solution: $\sin\theta = -\frac{1}{5}$

$reference \angle = \sin^{-1}\left(\frac{1}{5}\right) = 11{,}54°$

III: $\theta = 180° + 11{,}54° + k.360°$

$\boldsymbol{\theta = 191{,}54° + k.360°;\ k \in Z}$

IV: $\theta = 360° - 11{,}54° + k.360°$

$\boldsymbol{\theta = 348{,}46° + k.360°;\ k \in Z}$

The General Solution:

$\boldsymbol{\theta = 191{,}54° + k.360°}$ **or** $\boldsymbol{\theta = 348{,}46° + k.360°;}$ **where** $\boldsymbol{k \in Z}$

$\therefore \theta \in \{-168{,}46°;\ -11{,}54°;\ 191{,}54°;\ 348{,}46°\}$

Steps:

1) Calculate the reference angle ignoring the sign of the trig ratio.
3) sin is negative in quadrants 3 and 4. In

   Quadrant III: $\theta = 180° + ref\ \angle$

   Quadrant IV: $\theta = 360° - ref\ \angle$

4) Add $k.360°$, to determine the general solution.
5) To find specific solutions , we substitute integer values for k into the general solution arrived at.
5) Start with 0, 1, 2 and then -1, -2 etc. as value(s) for k, to calculate θ untill the angle is outside the given interval. See the calculations below:

$k = 0:\quad \theta = 191{,}54 + (0).360 = 191{,}54°$ ✓
$k = 1:\quad \theta = 191{,}54 + (1).360 = 551{,}54°$ ✗
$k = -1:\quad \theta = 191{,}54 + (-1).360 = -168{,}46°$ ✓
$k = -2:\quad \theta = 191{,}54 + (-2).360 = -528{,}46°$ ✗

$k = 0:\ \theta = 348{,}46° + (0).360 = 348{,}46°$ ✓
$k = 1:\ \theta = 348{,}46° + (1).360 = 708{,}46°$ ✗
$k = -1:\ \theta = 348{,}46° + (-1).360 = -11{,}54°$ ✓
$k = -2:\ \theta = 348{,}46° + (-2).360 = -371{,}54°$ ✗

CAN YOU:

Solve each of the following equations in the given interval:

1) $5 \sin\theta = 2;\qquad \theta \in [-360°\,;360°]$
2) $3 \tan\theta + 3 = 0\,;\quad \theta \in [-720°\,;0°]$

Answers:

1) $\theta \in \{-336{,}42°;\ -203{,}58°; 23{,}58°;\ 156{,}42°\}$
2) $\theta \in \{-585°;\ -405°; -225°;\ -45°\}$

Lesson 3 : Trig Equations requiring factorisation and applying Identities:

Some trigonometric equation could require factorization in order to solve it.
Check for the following types of **Factorisation**

- Common factor: $\sin x + 3\sin x.\cos x = 0$
- Difference of squares : $\sin^2 x - 9\cos^2 x = 0$
- Trinomial: $\sin^2 x - 2\cos x.\sin x + \cos^2 x = 0$

Example 5

Determine the general solution of $2\sin^2\theta = \sin\theta$

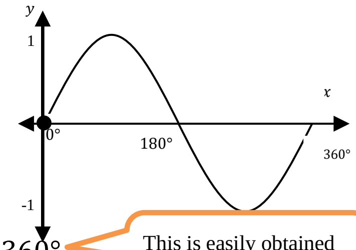
Solution: $2\sin^2\theta = \sin\theta$

$2\sin^2\theta - \sin\theta = 0$

$\sin\theta\,(2\sin\theta - 1) = 0$

$\therefore\ \sin\theta = 0 \quad \text{or} \quad 2\sin\theta - 1 = 0$

$\therefore\ \sin\theta = 0 \quad \text{or} \quad \sin\theta = \frac{1}{2}$



$\sin\theta = 0$

reference $\angle = \sin^{-1}(0) = 0°$

$\theta = 0° + k.360° \quad \text{or} \quad \theta = 180° + k.360°$

This is easily obtained from the above graph

Of

$\sin\theta = \frac{1}{2}$

Verwysings $\angle = \sin^{-1}\left(\frac{1}{2}\right) = 30°$

$\theta = 30° + k.360° \quad \text{or} \quad \theta = (180° - 30°) + k.360°$

$\therefore\ \theta = 30° + k.360° \quad \text{or} \quad \theta = 150° + k.360° \quad ; \ k \in Z$

$\therefore\ \sin\theta = 0 \ : \theta = 0° + k.360° \quad \text{or} \quad \theta = 180° + k.360° ; \ k \in Z$

$\therefore\ \sin\theta = \frac{1}{2} \ : \theta = 30° + k.360° \quad \text{or} \quad \theta = 150° + k.360° ; \ k \in Z$

Steps

1) Note that the equation is a quadratic, thus it can be written into the standard form of a quadratic equation. i.e $ax^2 + bx = 0$, as, $c = 0$.
2) Factorise – common factor
3) Recall: $A.B = 0$, then $A = 0$ or $B = 0$.
   This thus results in two trig equations that must be solved.
4) Calculate the reference angle.
5) Use the sign to determine in which quadrant sin is positive. It is the 1st and 2nd quadrant. Note that the solution of, $\sin\theta = 0$, can easily be determined from the sine graph. That is where the sine graph intersects the x-axis, at 0°, 180° and 360°. 360° is not required, because 0° + 360°.
6) Add $k.360°$, to determine the general solution.

Using Identities to solve equations: **1)** $\frac{\sin\theta}{\cos\theta} = \tan\theta$ **2)** $\sin^2\theta + \cos^2\theta = 1$

Example 6

Determine the general solution of $3\sin\theta + 4\cos\theta = 0$

Solution: $3\sin\theta + 4\cos\theta = 0$

$3\sin\theta = -4\cos\theta$

$\frac{3\sin\theta}{\cos\theta} = \frac{-4\cos\theta}{\cos\theta}$

$3\tan\theta = -4$

$\tan\theta = -\frac{4}{3}$

$reference\ \angle = \tan^{-1}\left(\frac{4}{3}\right) = 53{,}13°$

$\theta = 126{,}87° + k.180° ; \ k \in Z$

Answer:

$\theta = 126{,}87° + k.180° ; \ k \in Z$

OR

$\theta = 126{,}87° + k.360° \ \text{or}\ \theta = 306° + k.360° ; \ k \in Z$

Steps:

1) **Notice two different ratios but the same angle.**
2) Divide by $\cos\theta$ on both sides.
3) Use identity to simplify to one trig ratio.
4) Isolate the trig ratio.
5) Calculate the reference angle
6) Use the sign of trig ratio to determine in which quadrant tan is negative.
7) tan is negative in 2nd and 4th quadrant. In,
   Quadrant II: $\theta = 180° - ref\ \angle$
   Quadrant IV: $\theta = 360° - ref\ \angle$
   Add $k.360°$, to determine the general solution.
8) As the tan graph repeats itself every 180°, we need not consider the solution in the 4th quadrant. This is because the solution in the 2nd quadrant plus 180°, is the solution in the 4th quadrant.
9) Thus for **$\tan\theta$, only one solution in the interval [0°; 180°] is required then add $k.180°$ where $k \in Z$, for the general solution.**

10) Simplify

Example 7
Determine the general solution of $2\cos^2\theta = \sin\theta + 1$

Solution :

$$2\cos^2\theta - \sin\theta - 1 = 0$$
$$2(1 - \sin^2\theta) - \sin\theta - 1 = 0$$
$$2 - 2\sin^2\theta - \sin\theta - 1 = 0$$
$$2\sin^2\theta + \sin\theta - 1 = 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = -1$$

reference $\angle = \sin^{-1}\left(\frac{1}{2}\right) = 30°$

$\theta = 30° + k.360°$ of $\theta = (180° - 30°) + k.360°$

OF

reference $\angle = \sin^{-1}(1) = 90°$

$\theta = 180° + 90° + k.360° \quad ; k \in Z$

Answer:

$\theta = 30° + k.360°$ or $\theta = 150° + k.360°$
or $\theta = 270° + k.360° \quad ; k \in Z$

Steps

1) Using, Trig Identity, $\mathbf{\cos^2 x = 1 - \sin^2 x}$
2) Simplify.
3) Notice that this is a quadratics equation, it can thus be written into the form, $ax^2 + bx + c = 0$.
4) Factorise the quadratic trinomial
5) Recall: $A.B = 0$, then $A = 0$ of $B = 0$.
   This thus results in two trig equations that must be solved.
   Solve both trig equations separately
6) Isolate the trig ratio
7) Calculate the reference angle
8) Use the sign to determine in which quadrant the trig ratio is positive.
   sin is positive in 1st and 2nd quadrant
   And sin is negative in 3rd and 4th quadrant.

NB:
The equation, $\sin\theta = -1$, can be easily solved by using the sin graph. Look at the sketch of the sine graph given in the solution of example 5, and you will then see that, $\sin 270° = -1$

Example 8
Determine the general solution of $\cos\theta = -\cos 2\theta$

Solution: $\cos\theta = -\cos 2\theta$

$\cos\theta = \cos(180° - 2\theta)$

II $\theta = 180° - 2\theta + k.360° ; \quad k \in Z$

$\therefore 3\theta = 180° + k.360°$

$\therefore \theta = 60° + k.120°$

III $\cos\theta = \cos(180° + 2\theta)$

$\theta = 180° + 2\theta + k.360° ; \quad k \in Z$

$\therefore -\theta = 180° + k.360°$

$\therefore \theta = -180° - k.360°$

Answer:
$\theta = 60° + k.120°$ or $\theta = -180° - k.360° \; ; \; k \in Z$

Steps

1) **Notice same ratio but different angle**
2) Recall from CAST diagram:
   $\cos(180° - \theta) = -\cos\theta$
   $\cos(180° + \theta) = -\cos\theta$
   $\therefore \; -\cos 2\theta = \cos(180° - 2\theta)$
   $\therefore \; -\cos 2\theta = \cos(180° + 2\theta)$
3. Note: if $\cos A = \cos B$ then
   $A = B + k.360°$

CAN YOU : Solve the following equations:

1) $5\cos^2\theta.\cos\theta = 0$
2) $5\sin A = 3\cos A$
3) $5 - 4\sin^2\theta + 4\cos\theta = 0$
4) $\sin 2\theta = \sin(\theta - 30°)$

Answers:

1) $\theta = 90° + k.360°$ or $\theta = 270° + k.360°; k \in Z$
2) $A = 30{,}96° + k.360°$ or
   $A = 210° + k.360°; \quad k \in Z$
3) $\theta = 120° + k.360°$ or $\theta = 240° + k.360°; k \in Z$
4) $\theta = -30° + k.360°$ or $\theta = 70° + k.120°; k \in Z$

Example 9 Co-Functions
Determine the general solution of
$\sin(\theta + 20°) = \cos 2\theta$

Solution: $\sin(\theta + 20°) = \cos 2\theta$

$\sin(\theta + 20°) = \sin(90° - 2\theta)$

Ref angle $= (90° - 2\theta)$

Steps

1) **Notice different ratio and different angle**
2) **Co-functions**: replace $\cos 2\theta = \sin(90° - 2\theta)$
3) Ratio is the same and angles is different.
4) Use the sign to determine in which quadrant sin is positive i.e. 1st and 2nd quadrant
5) Solve θ

I : $\theta + 20° = 90° - 2\theta + k.360°$ $3\theta = 70° + k.360°$ $\theta = 23{,}33° + k.120°;\ k \in Z$ **II** : $\theta + 20° = 180° - (90° - 2\theta) + k.360°$ $-\theta = 70° + k.360°$ $\theta = -70° - k.360°;\ k \in Z$ **Answer:** $\theta = 23{,}33° + k.120°$ or $\theta = -70° - k.360°;\ k \in Z$	

Lesson 4:

An Identity is undefined:

1) when **any denominator equals zero, i.e division by zero,**

2) if $\theta = 90° + k.180°$; $k \in$ Z and the identity contains $\tan\theta$. This is because tan 90°, is undefined.

Example 10	**Solution:**	**Steps:**
Determine the values of θ for which the identity $\frac{\tan\theta + \sin\theta}{1 + \frac{1}{\cos\theta}} = \sin\theta$ is undefined	$\theta = 90° + k.180°$; $k \in$ Z or $1 + \frac{1}{\cos\theta} = 0$ $\frac{1}{\cos\theta} = -1$ $\therefore \cos\theta = -1$ $\theta = 180° + k.360°;\ k \in Z$ **Answer:** $\theta = 90° + k.180°$ or $\theta = 180° + k.360°;\ k \in Z$	1) $\tan\theta$ 2) when denominator equals zero

CAN YOU : Determine the values of θ for which the identity is not defined 1) $\frac{1}{\sin\theta} - \sin\theta = \frac{\cos\theta}{\tan\theta}$ 2) $\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = \frac{2}{\cos^2\theta}$	**Answers:** 1) $\theta = 0° + k.180°$ or $\theta = 90° + k.180°$; $k \in Z$ 2) $\theta = 90° + k.180°$; $k \in Z$

	Siyavula	Mind Action series	Platinum	Via Afrika
ACTIVITIES/ ASSESSMENT	Exer 6-7,Pg 269 Exer 6-8, Pg 275 Exer 6-9 , Pg 280	Exer 9- Pg 131 Exer 10- Pg 133 Exer 11- Pg 137 Exer 12- Pg 139 Exer 13- Pg 140	Exer 15- Pg 155 Exer 16- Pg 157 Exer 17- Pg 158 Exer 18- Pg 159 Exer 19&20- Pg 161 Exer 21- Pg 162	Exer 16, Pg 179 Exer 17, Pg 180 Exer18& 19, Pg 181 Exer20& 21, Pg 183
CONSOLIDATION	• Isolate the Trigonometric ratio(standard form) : **Trig ratio (angle) = number** • Determine the reference angle • Use CAST-diagram to determine in which quadrant the angle lies. • Special cases: 1) more than one different ratio 2) Identities 3) Same ratio but different angle 4) For a given interval			