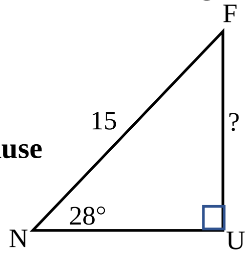
Western Cape Government
Education

Directorate: Curriculum FET

SUBJECT and GRADE	Mathematics Gr 11	
TERM 3	Week 5	
TOPIC	Trigonometry: Formulae to solve triangles	
AIMS OF LESSON	To introduce learners to formulae to calculate the sizes of angles and lengths of sides of any triangle.	
RESOURCES	***Paper based resources***	***Digital resources***
	Please refer to the Chapter on Trigonometry for the section on Sine and Cosine Rule in your Textbook.	*Sine Rule:* https://www.youtube.com/watch?v=r4YJuhS-1XE
INTRODUCTION	• In Gr 10 you were introduced to Trigonometry where we used trigonometric ratios (sin θ, cos θ and tan θ) and Pythagoras to calculate the angles and sides of a **right-angled triangle.** Otherwise we drew a right-angled triangle in the Cartesian plane to solve the angles and sides through trig ratios and using Pythagoras. • In the following lessons for Gr 11 we will introduce 3 rules/ formulae (Sine rule, Cos rule and Area rule) that will enable us to calculate the sides and angles of any triangle, right angled or not, as well as calculate the area of any triangle. • These rules are very important for further study of Trigonometry and you are required to KNOW THE PROOFS of these rules for examination purposes. It forms part of the 12 marks for Bookwork that will be tested in Paper 2.	
CONCEPTS AND SKILLS	• Using Theorem of Pythagoras • Solving right-angled triangles • Drawing Perpendicular heights of triangles • Using trig ratios: sin θ, cos θ and tan θ • Using the scientific calculator to do trigonometric calculations	

Lesson 1: Revising Gr 10 Trigonometry: Solving Triangles

1. In ΔFUN, FN = 15, ∠ U = 90° and ∠ N = 28°. Calculate the length of UF



Solution:

Side UF is **Opposite** to 28° and FN is the **Hypotenuse**

Write ratio with UF as numerator

$\frac{UF}{15} = \frac{o}{h} = \sin 28°$

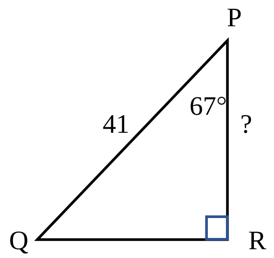
$\therefore UF = 15 \sin 28°$ [1] [5] [sin] [2] [8] [)] [=]

$= 7{,}04$

***Note**: We can use the Theorem of Pythagoras or cos 28° to calculate the other side.*

CAN YOU?

1. In ΔPQR, PQ = 41, ∠R = 90° and ∠P = 67°. Calculate the length of PR.

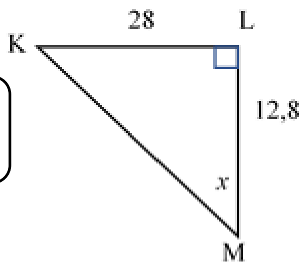


Solution: PR = 16,02

2. Calculate the size of angle x in the diagram

28 is **opposite** to x and 12,8 is **adjacent** to x

Choose **tan x** since **opp** and **adj** sides are given



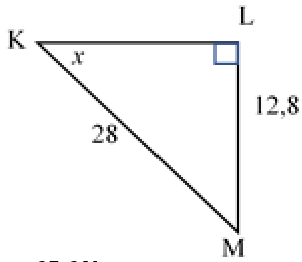
$\Rightarrow \tan x = \frac{o}{a} = \frac{28}{12{,}8} = 2{,}1875$

$\therefore x = \tan^{-1}(2{,}1875)$ SHIFT tan 2 • 1 8 7 5) =

$= 65{,}43°$

CAN YOU?

2. Calculate the size of angle x in the diagram

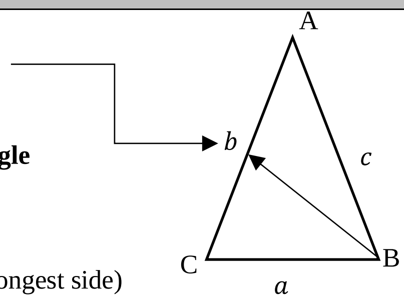


Solution: x = 27, 20°

Lesson 2: The **sine rule or just (sin rule)**

Note:

- We use small caps to denote the side **opposite** a vertex
- In a Δ we get the **shortest side** opposite the **smallest angle** and vice versa
- There can only be **1 obtuse angle** in a Δ (opposite the longest side)



Sin-rule: In any ΔABC we have that: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ OR $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Proof : Given any ΔABC. Proof that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ [We will do this for when $\hat{A}$ is acute and for when $\hat{A}$ is obtuse]
(Note: Proof might be different in your textbook)

Construction: Draw ΔABC with h the perpendicular height from B onto AC (fig. 1) or AC produced (fig. 2)

Fig.1

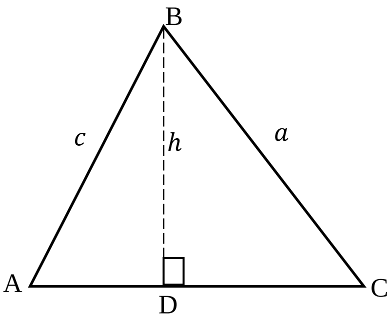
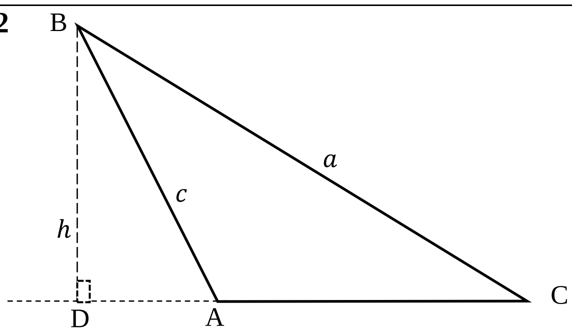


Fig. 2



$\sin A = \frac{h}{c}$ [in ΔABD] and $\sin C = \frac{h}{a}$ [in ΔCBD] [$\because \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$]

$\therefore\ h = c\sin A$ and $h = a\sin C$

$\therefore c\sin A = a\sin C \Rightarrow \therefore \frac{c\sin A}{ac} = \frac{a\sin C}{ac}$

$\therefore \frac{\sin A}{a} = \frac{\sin C}{c}$

Similarly, by constructing a ⊥height from C onto AB, we can proof that: $\frac{\sin A}{a} = \frac{\sin B}{b}$

$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

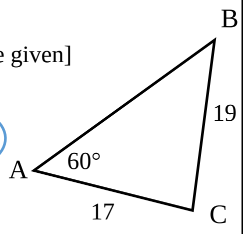
The sine-rule is usually used when **2 sides and an angle,** *opposite* 1 of the 2 sides, are given or when **2 angles and a side** are given as we will show in the next 2 examples

Examples: Applying the sin-rule

1. In ΔABC we have AC = 17, BC = 19 and $B\hat{A}C = 60°$
   Calculate the size of $\hat{B}$

Solution:

[2 sides and an angle opposite 1 of the sides are given]



Write the sin-rule with sin B as numerator

$\therefore \frac{\sin B}{b} = \frac{\sin A}{a}$

BC = a and AC = b

$\therefore \frac{\sin B}{17} = \frac{\sin 60°}{19}$

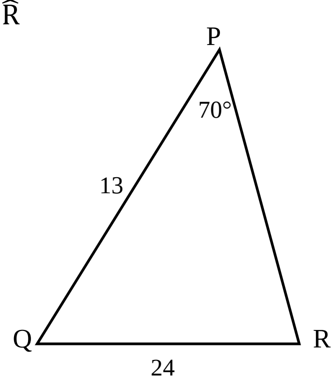
$\therefore \sin B = \frac{17 \sin 60°}{19} = 0{,}7748..$

$\hat{B}$ must be less than 60° Why?

$\therefore \hat{B} = 50{,}79°$

Can you do?

1. In ΔPQR we have PQ = 13, QR = 24 and $\hat{P} = 70°$
   Calculate the size of $\hat{R}$

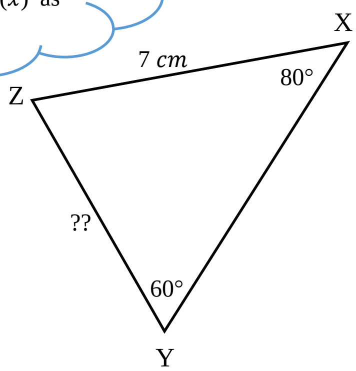


Answer: $\hat{R} = 30{,}6°$

2. Calculate the length of YZ in the following diagram:

Solution: [2 angles and a side are given]

Write the sin-rule with side YZ (x) as numerator



$\therefore \frac{x}{\sin X} = \frac{y}{\sin Y}$

$\therefore \frac{x}{\sin 80°} = \frac{7}{\sin 60°}$

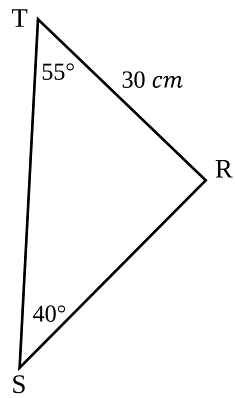
$\therefore x = \frac{7 \sin 80°}{\sin 60°}$

$= 7{,}96\ cm$

Note: $\hat{Z}$ *can be calculated through sum of angles of a Δ and then the side XY again using sin-rule*

Can you do?

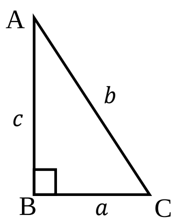
2. Calculate the length of RS in the following diagram:



Answer: RS = 38,23 cm

Note: If we have a right-angled Δ, applying the sin-rule leads to the sine-ratio

From the diagram:



$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\therefore \frac{\sin A}{a} = \frac{\sin 90°}{b}$

sin 90°=1

$\therefore \sin A = \frac{a \sin 90°}{b} = \frac{a(1)}{b} = \frac{a}{b} = \frac{opp}{hyp}$

Lesson 3: sine-rule (cont.)

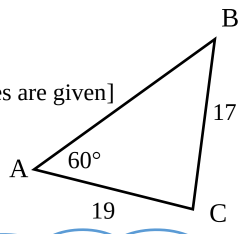
The ambiguous case: [2 sides and an angle given]
As mentioned, we get the smallest angle opposite the shortest side, and hence the biggest angle will be opposite the longest side of a Δ.

Thus, if we are ask to calculate the size of an angle opposite the longer of the 2 given sides, we can get 2 values (1 acute and the other obtuse) for that angle, unless stated otherwise.

We will illustrate this by changing example 1 accordingly:

3. In ΔABC we have AC = 19, BC = 17 and $B\hat{A}C = 60°$
   Calculate the size of $\hat{B}$

Solution:
[2 sides and an angle opposite 1 of the sides are given]



$\therefore \frac{\sin B}{b} = \frac{\sin A}{a}$

$\therefore \frac{\sin B}{19} = \frac{\sin 60°}{17}$

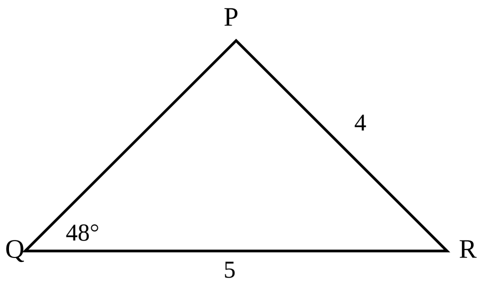
Since AC = 19 is longer than BC, we can get 2 values for $\hat{B}$ (> 60°)

$\therefore \sin B = \frac{19 \sin 60°}{17} = 0{,}967910\ldots..$

$\therefore$ **$\hat{B}$ = 75,45°** OR **$\hat{B}$** = 180° - 75,45° = **104,55°**

Can you do?

3. Calculate the size of $\hat{P}$ if QR = 5, PR = 4, $\hat{Q} = 48°$ and $\hat{P}$ is obtuse.



Answer: $\hat{P} = 111{,}73°$

Exercises on the sine-rule from your textbook.

Lesson 4: The cosine-(cos) rule

If **3 sides** are given, we **cannot use the sine-rule**, since we need at least 1 angle.

The same thing will happen if we are given **2 sides and the included angle**, since 1 of the angles must be opposite 1 of the given sides. (Angle between the 2 sides)

To overcome this, we introduce the cos-rule

Cos-rule: In any ΔABC we have that:

$a^2 = b^2 + c^2 - 2bc\cos A$ OR

$b^2 = a^2 + c^2 - 2ac\cos B$ OR

$c^2 = a^2 + b^2 - 2ab\cos C$

(For calculating **sides**)

OR

$\cos A = \frac{b^2+c^2-a^2}{2bc}$

$\cos B = \frac{a^2+c^2-b^2}{2ac}$

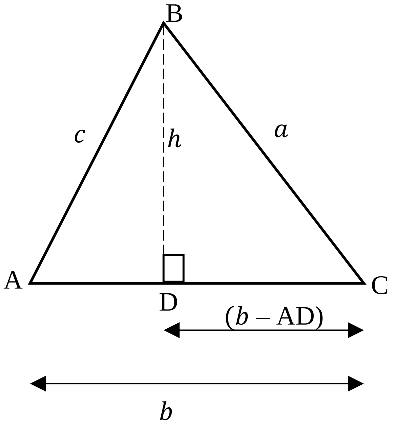
$\cos C = \frac{a^2+b^2-c^2}{2ab}$

(For calculating **angles**)

Proof : Given any ΔABC. Proof that $a^2 = b^2 + c^2 - 2bc\cos A$ [We will do this for when Â is acute and obtuse]
(Note: Proof might be different in your textbook)

Construction: Draw ΔABC with h the perpendicular height from B onto AC (fig. 1) or AC produced (fig. 2)

Fig.1



In ΔBDC: $a^2 = \text{DC}^2 + h^2$ Pyth

$= (b - \text{AD})^2 + h^2$

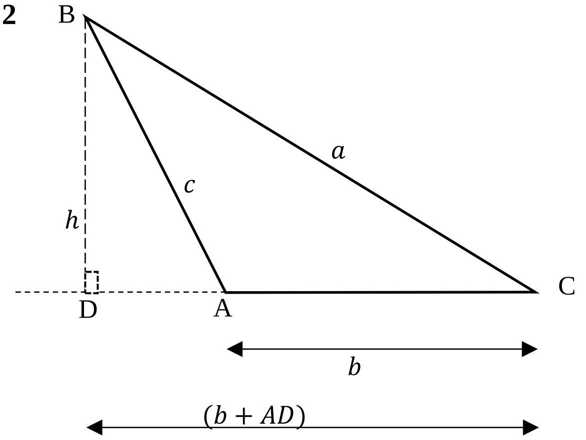
$= b^2 - 2b.\text{AD} + AD^2 + h^2$

$= b^2 - 2b.\text{AD} + c^2$ ... $[AD^2 + h^2 = c^2]$ Pyth

But in ΔABD: $\frac{\text{AD}}{c} = \sin A \Rightarrow \text{AD} = c\sin A$

$\therefore a^2 = b^2 + c^2 - 2bc\cos A$

Fig. 2



In ΔBDC: $a^2 = \text{DC}^2 + h^2$ Pyth

$= (b + \text{AD})^2 + h^2$

$= b^2 + 2b.\text{AD} + AD^2 + h^2$

$= b^2 + 2b.\text{AD} + c^2$ ... $[AD^2 + h^2 = c^2]$ Pyth

But in ΔABD: $\frac{\text{AD}}{c} = \sin(180° - A) = -\sin A$

$\Rightarrow \text{AD} = -c\sin A$

$\therefore a^2 = b^2 + c^2 - 2bc\cos A$

Lesson 5: Application of the cos-rule

Note: The cos rule is used when:

- 3 sides are given (and you have to calculate an angle)
- 2 sides and the included angle given (and you have to calculate the 3rd side)

Example 1.

In Δ DEF we have $d = 4$; $f = 7$ and $\hat{\mathrm{E}} = 65°$.

Calculate the length of DF(e)

Solution: [2 sides and inlcuded angle given]

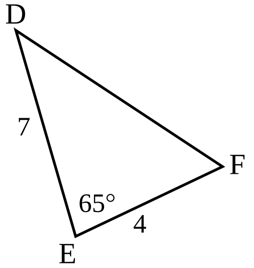
$$e^2 = d^2 + f^2 - 2df \cos \mathrm{E}$$

Cos-rule for ΔDEF to calculate **side *e***

$$\therefore e^2 = (4)^2 + (7)^2 - 2(4)(7)\cos 65°$$

$$\therefore e^2 = 41{,}33$$

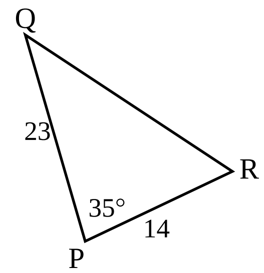
$$\therefore \mathrm{DF} = \sqrt{41{,}33} = 6{,}43$$



Can you do?

In Δ QPR we have $q = 14$; $r = 23$ and $\hat{\mathrm{P}} = 35°$.

Calculate the length of QR



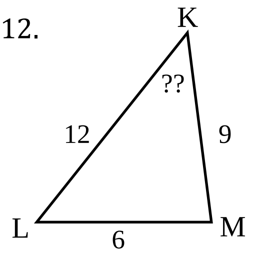
Answer: QR = 14,05

Example 2.

In ΔKLM, $k = 6$; $l = 9$ and $m = 12$.

(a) Calculate the size of $\hat{\mathrm{K}}$

(b) Calculate the size of $\hat{\mathrm{M}}$



Solution:

(a) [3 sides are given]

$$k^2 = l^2 + m^2 - 2lm \cos \mathrm{K}$$

$$\therefore \cos \mathrm{K} = \frac{l^2 + m^2 - k^2}{2lm}$$

Cos-rule for ΔKLM to calculate **angle K**

$$\therefore \cos \mathrm{K} = \frac{(9)^2 + (12)^2 - (6)^2}{2(9)(12)} = 0{,}875$$

$$\therefore \hat{\mathrm{K}} = 28{,}96°$$

(b) Since an angle + (at least) 2sides with a side opposite the angle are given, use **sin-rule** further!!

$$\therefore \frac{\sin \mathrm{M}}{m} = \frac{\sin \mathrm{K}}{k}$$

$$\therefore \frac{\sin \mathrm{M}}{12} = \frac{\sin 28{,}96°}{6}$$

$$\therefore \sin \mathrm{M} = \frac{12 \sin 28{,}96°}{6} = 0{,}968 \ldots$$

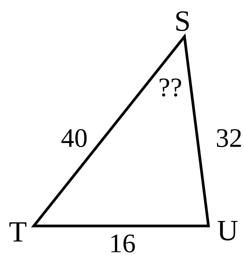
$$\therefore \mathrm{M} = 75{,}53°$$

Can you do?

In ΔSTU, $s = 16$; $t = 32$ and $u = 40$.

(c) Calculate the size of $\hat{\mathrm{S}}$

(d) Calculate the size of $\hat{\mathrm{T}}$



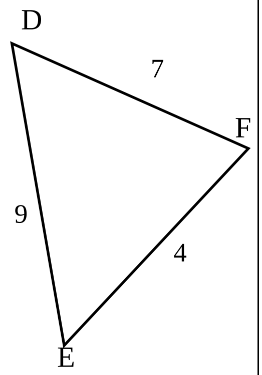
Answer:

(a) $\hat{\mathrm{S}} = 22{,}33°$

(b) $\hat{\mathrm{T}} = 49{,}45°$

Example 3.

Calculate the size of $\hat{F}$ using the diagram:



Solution: [3 sides given: use cos-rule]

$$\therefore \cos F = \frac{d^2 + e^2 - f^2}{2de}$$

$$= \frac{(4)^2+(7)^2-(9)^2}{2(4)(7)}$$

$$= -0{,}285 \ldots$$

(−) value shows that $\hat{F}$ is ***obtuse***, hence subtract ref. ∠ from 180°

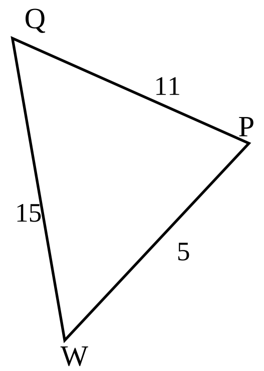
Ref. ∠: 0,285 cos^{-1} = 73,44°

$\therefore \hat{F} = 180° - 73{,}44°$

= 106, 56°

Can you do?

Calculate the size of $\hat{P}$ using the diagram:



Answer: $\hat{P}$ = 126, 76°

ACTIVITIES/ ASSESSMENT	*Please do the exercises as they appear in you textbook at the end of the* ***sine rule*** *as well as those after the* ***cos rule***

CONSOLIDATION:	• Sin-rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ OR $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ ✓ Used when **2 sides and an angle,** *opposite* 1 of the 2 sides, are given or when **2 angles and a side** are given • Cos-rule: $a^2 = b^2 + c^2 - 2bc \cos A$ to calculate a side OR $\cos A = \frac{b^2+c^2-a^2}{2bc}$ for calculating a side. ✓ Used when **2 sides and** the include **angle,** or ewhen 3 sides are given • The cos-rule are used once in a triangle and thereafter use sin-rule to calculate other angles/ sides. • In the next lesson we will look at the Area-rule which will enable us to calculate the area for any triangle. • Thank you for participating in this lesson and please continue to work through the lessons as they are made available. • Remember: Your hard work will reap success at the end!! KEEP IT UP!!