SUBJECT and GRADE	Mathematics Grade 12	
TERM 1	Week 5	
TOPIC	Euclidean Geometry - Similarity Theorem	
AIMS OF LESSON	Equiangular triangles are similar If the corresponding sides of two triangles are in the same proportion, then the triangles are similar	
RESOURCES	***Paper based resources***	***Digital resources***
	Mind the Gap; Your textbook	https://www.youtube.com/watch?v=VZt4wXBo1PA

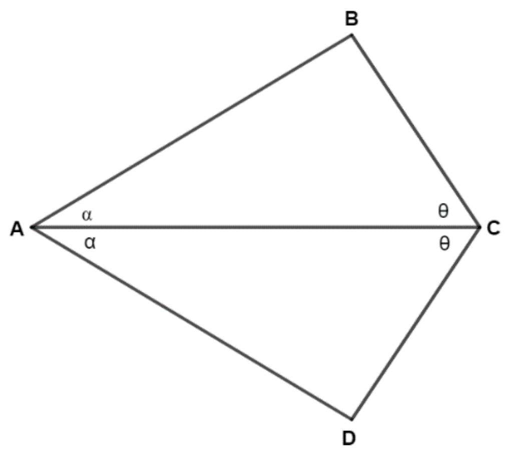
INTRODUCTION

Let's go back to the previous grades and review congruency for triangles.

<u>**CONGRUENCY (≡):**</u> Two triangles is **congruen**t if they have **the same shape and size.** Thus the two triangles are **identical**, the **corresponding angles are the same** and **the corresponding sides are equal**.

Two triangles are congruent if they have:

1. 3 sides the same length: (S; S; S)
2. 2 sides and an included angle: (S; A; S)
3. 2 angles and a side equal: (A; A; S)
4. A right angle, hypotenuse and a side equal: (R; H; S)



In ΔABC and ΔADC :

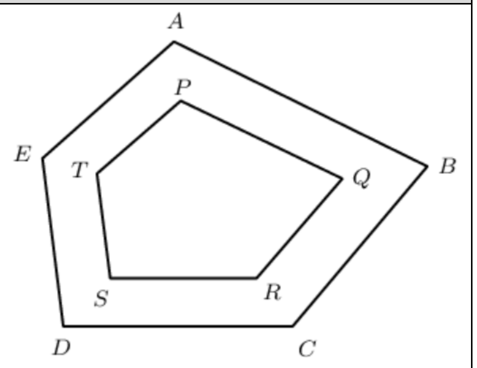
1. $B\hat{A}C = D\hat{A}C$ given
2. $B\hat{C}A = D\hat{C}A$ given
3. AC is common

$\therefore \Delta ABC \equiv \Delta ADC$ (A; A; S)

Since they are congruent, we can say now that: $\hat{B} = \hat{D}$; $AB = AD$ and $BC = DC$

CONCEPTS AND SKILLS

SIMILARITY (|||): Polygons are **similar** if they have the **same shape**. If two polygons are similar, the one is an enlargement of the other.



Two polygons are similar if and only if:

All pairs of corresponding angles are equal **AND**

all pairs of corresponding sides are in the same proportion.

Both of these conditions have to be met for two polygons to be similar.

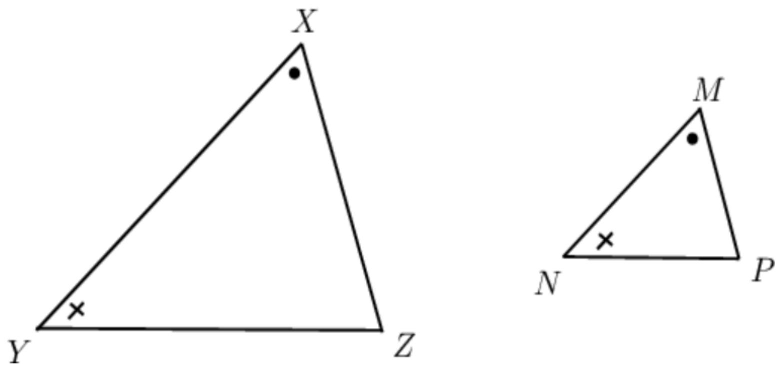
For Triangles, any one of these two conditions is sufficient to guarantee similarity.

Hence in any two triangles if:

All pairs of corresponding angles are equal, then the two triangles are similar,

OR

If all pairs of corresponding sides are in the same proportion, then the two triangles are similar.



If **one** of the **conditions is true** for two triangles, then **the other condition is automatically also true.**

NOTATION

$\Delta XYZ \;|||\; \Delta\, MNP$ means 'triangle XYZ is similar to triangle MNP'. The **order** in which the letters are written is very important, as it indicates **which angles are equal.**

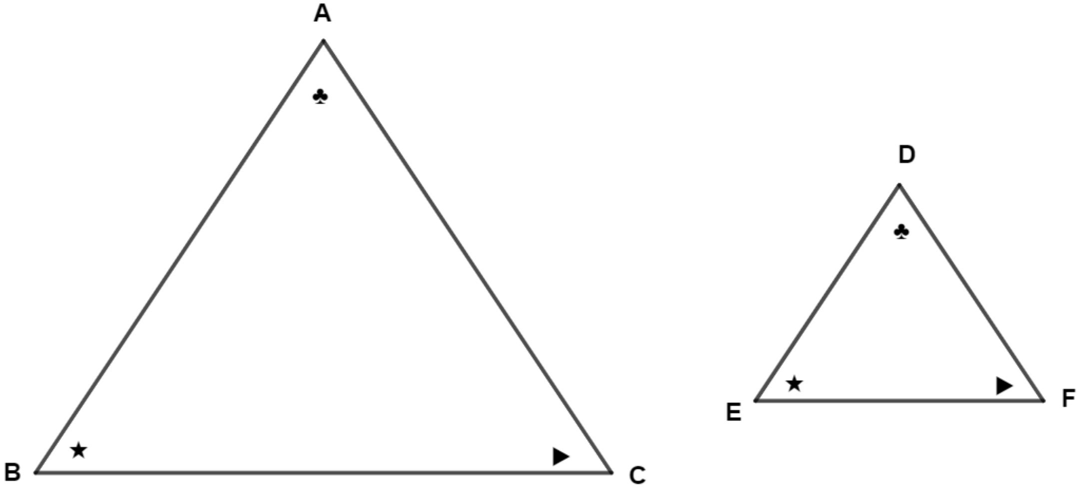
Hence $\hat{X} = \hat{M}$, $\hat{Y} = \hat{N}$ and $\hat{Z} = \hat{P}$.

The order also indicates which **ratios of sides are equal**: ΔXYZ ||| Δ MNP

$$\frac{XY}{MN} = \frac{YZ}{NP} = \frac{XZ}{MP} \quad \text{or} \quad \frac{XY}{YZ} = \frac{MN}{NP} \quad \text{or} \quad \frac{XY}{XZ} = \frac{MN}{MP} \quad \text{or} \quad \frac{YZ}{XZ} = \frac{NP}{MP}$$

<u>THEOREM 2</u>: TRIANGLE SIMILARITY THEOREM

If two triangles are equiangular, then their corresponding sides are in the same proportion and hence the triangles are similar.

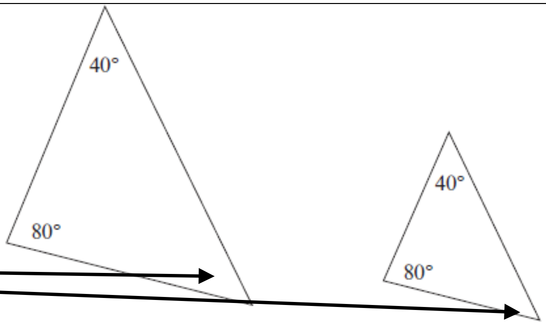


Given: $\Delta\mathbf{ABC}$ and $\Delta\mathbf{DEF}$ with $\hat{\mathbf{A}} = \hat{\mathbf{D}}$, $\hat{\mathbf{B}} = \hat{\mathbf{E}}$ and $\hat{\mathbf{C}} = \hat{\mathbf{F}}$

Conclusion: $\frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{AC}}{\mathbf{DF}} = \frac{\mathbf{BC}}{\mathbf{EF}}$ and hence $\Delta\mathbf{ABC}\ |||\ \Delta\ \mathbf{DEF}$. Reason: ∠∠∠

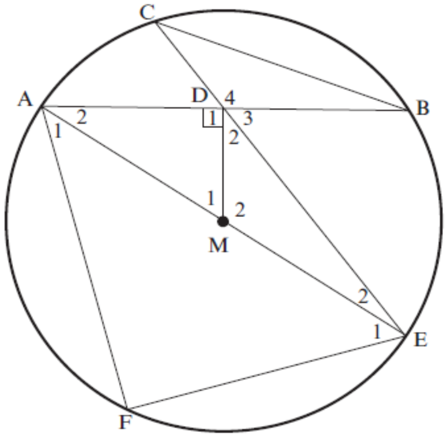
NOTE:
If two triangles have 2 corresponding angles equal, then the third angle in each triangle will equal each other (sum angles of a triangle = 180°) and the triangles are therefore similar and their sides will be in proportion. The shortened reason you can use is *(third angle)*

If **two angles are the same, then the 3rd angle of both triangles is**
180° – (40° + 80°) (sum angles in ∆) = 60°



Example 1:

Diameter AME of circle with centre M bisects $F\hat{A}B$
MD is perpendicular to the chord AB.
ED produced meets the circle at C, and CB is joined.



a) Prove $\Delta AEF \parallel\!| \Delta AMD$ (5)

b) Hence, find the numerical value of $\dfrac{AF}{AD}$ (5)

c) Prove $\Delta CDB \parallel\!| \Delta ADE$ (4)

d) Prove $AD^2 = CD.DE$ (3)

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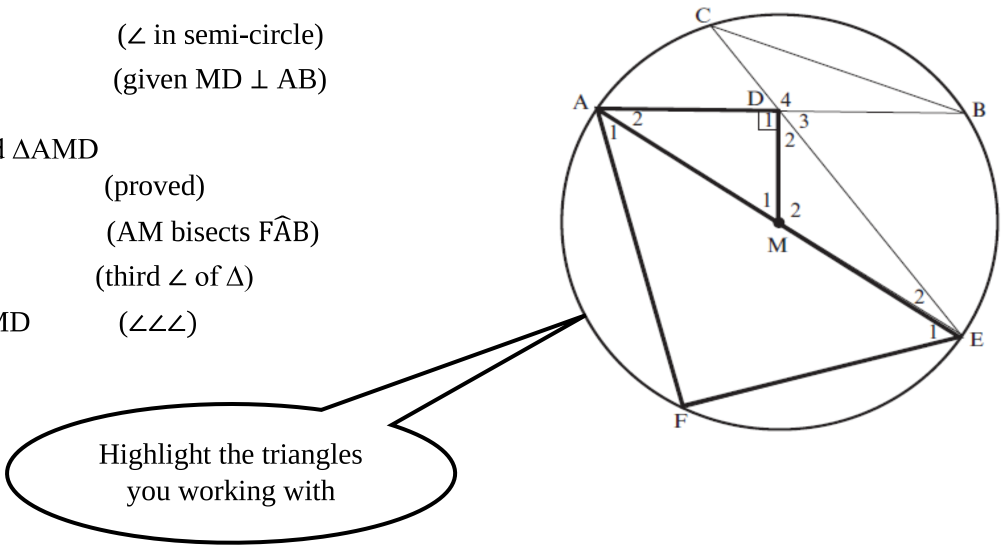
Solution:

a)
$\hat{F} = 90°$ (∠ in semi-circle)
$\hat{D}_1 = 90°$ (given MD ⊥ AB)
$\therefore \hat{F} = \hat{D}_1$
In ΔAEF and ΔAMD
$\hat{F} = \hat{D}_1$ (proved)
$\hat{A}_1 = \hat{A}_2$ (AM bisects $F\hat{A}B$)
$\therefore \hat{E}_1 = \hat{M}_1$ (third ∠ of Δ)
$\therefore \Delta AEF \parallel\!| \Delta AMD$ (∠∠∠)



b)
$\dfrac{AE}{AM} = \dfrac{EF}{MD} = \dfrac{AF}{AD}$ (||| Δs)

$AM = ME$ (radii)
$\therefore AE = 2AM$

$\therefore \dfrac{2AM}{AM} = \dfrac{AF}{AD}$

$\dfrac{AF}{AD} = 2$

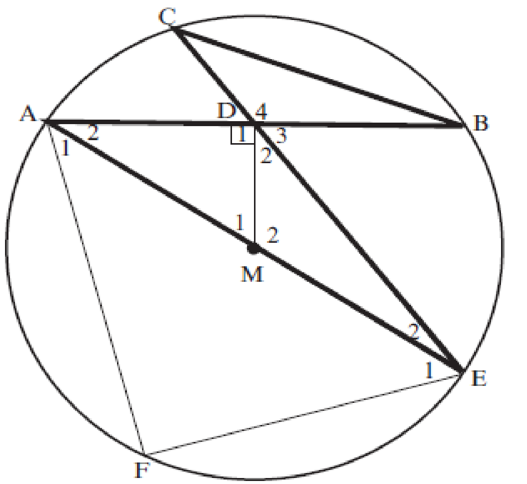
c) In ΔCDB and ΔADE

$\hat{C} = \hat{A}_2$ (∠s in same seg)

$\hat{B} = \hat{E}_2$ (∠s in same seg)

$\hat{D}_4 = \hat{D}_1 + \hat{D}_2$ (vert opp. ∠ s =)

∴ ΔCDB ||| ΔADE (∠∠∠)



d) $\frac{CD}{AD} = \frac{DB}{DE}$ (||| Δs)

∴ CD. DE = AD. DB

But AD = DB (MD ⊥ AB, M is centre)

∴ CD. DE = AD.AD

∴ AD^2 = CD. DE

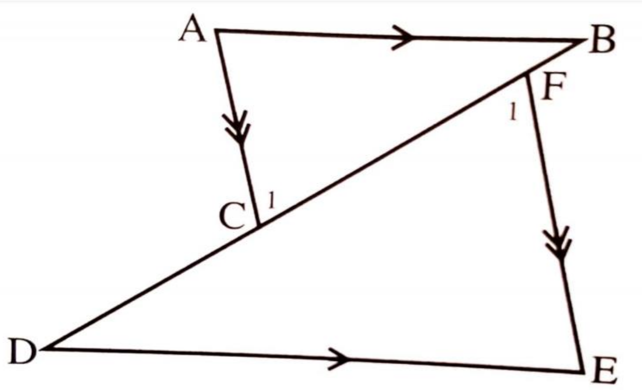
CAN YOU:

1) In the sketch below, AB ∥ DE and AC ∥ FE .

Prove that

(a) ΔBCA ||| ΔDFE

(b) AB. EF = AC. ED



2) In the sketch below, SR is a tangent to circle PST at S.

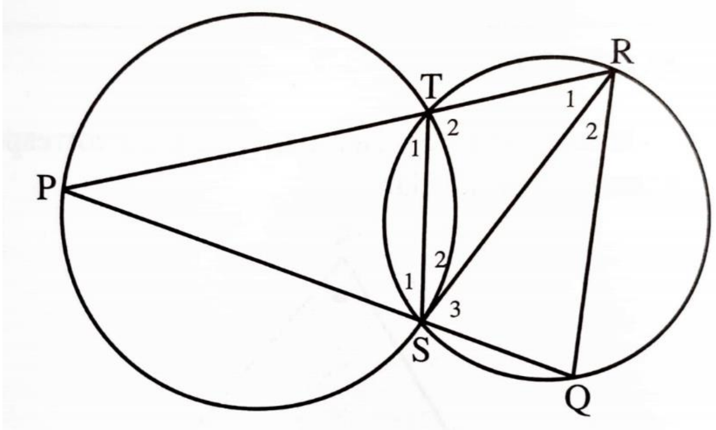
Prove that

(a) ΔPRS ||| ΔSRT

(b) RS^2 = PR. RT

(c) ΔPQR ||| ΔPTS

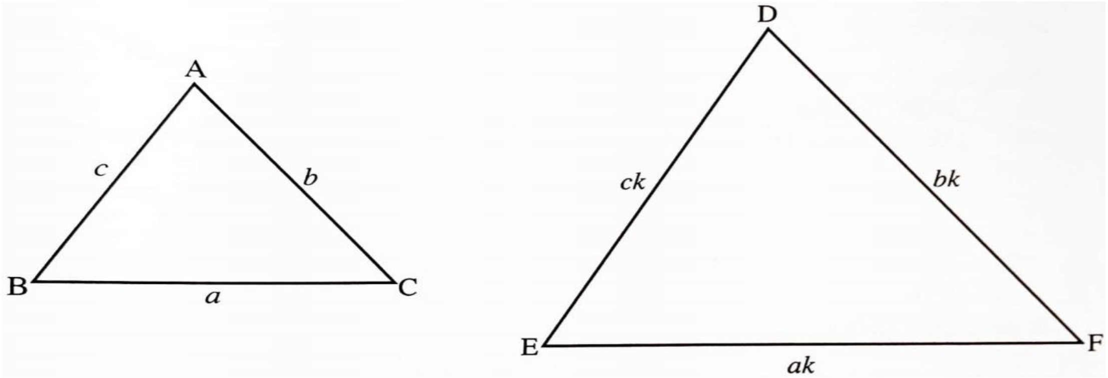
(d) $\frac{RT}{PT} = \frac{RS^2}{PQ\ .\ PS}$



Using sides to prove that triangles are similar

CONVERSE OF THEOREM 2.

If the corresponding sides of two triangles are in the same proportion, then the triangles are equiangular and hence similar.



Given: $\Delta\mathbf{ABC}$ and $\Delta\mathbf{DEF}$ with

$$\frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{AC}}{\mathbf{DF}} = \frac{\mathbf{BC}}{\mathbf{EF}}$$

Conclusion: $\hat{\mathbf{A}} = \hat{\mathbf{D}}$, $\hat{\mathbf{B}} = \hat{\mathbf{E}}$, $\hat{\mathbf{C}} = \hat{\mathbf{F}}$ and hence $\Delta\mathbf{ABC} \,|||\, \Delta\,\mathbf{DEF}$. **Reason**: sides of Δs in prop.

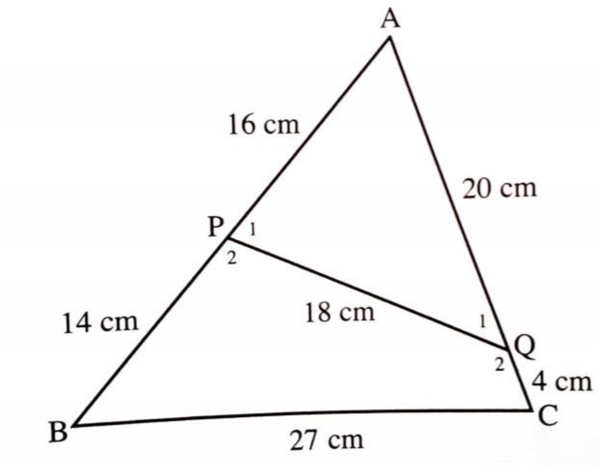
Example 2:

In the sketch alongside, AP = 16cm, PB = 14cm
AQ = 20cm, QC= 4cm and BC = 27cm.

Prove that :

(a) ΔAPQ ||| Δ ACB

(b) PBCQ is a cyclic quadrilateral.



Solution:

a) $\frac{\text{AP}}{\text{AC}} = \frac{16\text{cm}}{24\text{cm}} = \frac{2}{3}$

$\frac{\text{PQ}}{\text{CB}} = \frac{18\text{cm}}{27\text{cm}} = \frac{2}{3}$

$\frac{\text{AQ}}{\text{AB}} = \frac{20\text{cm}}{30\text{cm}} = \frac{2}{3}$

$\therefore \frac{\text{AP}}{\text{AC}} = \frac{\text{PQ}}{\text{CB}} = \frac{AQ}{\text{AB}}$

∴ ΔAPQ ||| Δ ACB (sides of Δs in prop.)

(b) $\hat{P}_1 = \hat{C}$ (||| Δs)

∴ PBCQ is a cyclic quad. (ext ∠ of quad = int opp∠)

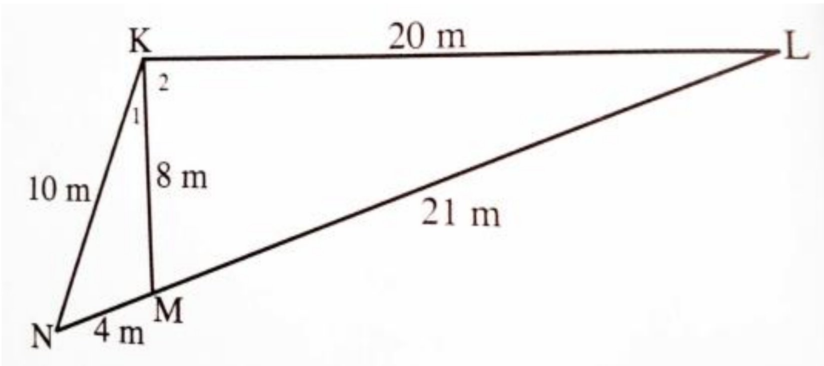
CAN YOU:

In the sketch alongside, KL= 20m, KN = 10m
MN = 4m, KM = 8m and LM= 21m.

Prove that :

(a) ΔKMN ||| Δ LKN

(b) KN is a tangent to circle LMK at K.



<u>Identifying Triangles</u>:

We are sometimes required to prove the equality of ratios and/or products, where the question doesn't state which triangles to prove similar.

In such cases we identify the triangles first.

Suppose, for example, you have to prove that $\frac{AB}{AC} = \frac{BC}{CD}$. There are 2 possible ways of identifying triangles in order to prove the ratios equal:

1) Top triangle, bottom triangle

ΔABC

$\frac{AB}{AD} = \frac{BC}{CD}$

ΔACD

Are the top sides (AB and BC) sides of one triangle

And the bottom sides (AD and CD) sides of another triangle

If each of these pairs are sides of a triangle in the sketch, then you can proceed to try proving these triangles similar

2) Left triangle, Right triangle

Are the left sides (AB and AD) sides of one

ΔABD | $\frac{AB}{AD} = \frac{BC}{CD}$ | ΔBCD

The right sides (BC and CD) sides of another triangle

If each of these pairs are sides of a triangle in the sketch, then you can proceed to try proving these triangles similar.

Example 3:

In the sketch alongside, AP is a tangent to the circle at P. PN ∥ SR

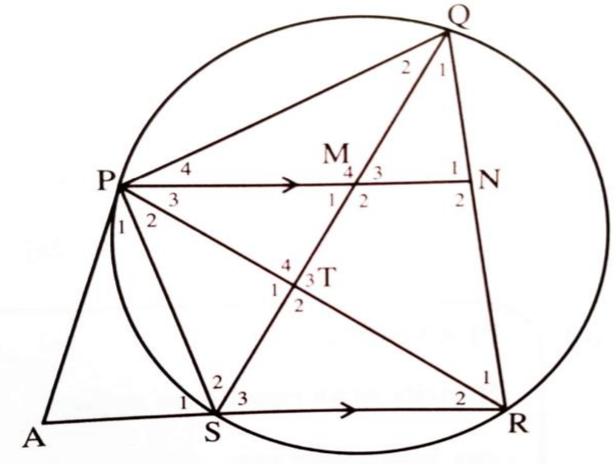
Prove that :

(a) $\frac{\text{PS}}{\text{QR}} = \frac{\text{ST}}{\text{RT}}$

(b) $\frac{\text{PQ}}{\text{PT}} = \frac{\text{SR}}{\text{ST}}$

(c) $\text{PM} \,.\, ST = \text{RS} \,.\, \text{MT}$

(d) $\text{AP}^2 = \text{AR} \,.\, \text{AS}$



Solution:

a)

ΔPST ✓

$\frac{\text{PS}}{\text{QR}} = \frac{\text{ST}}{\text{RT}}$

ΔQRT ✓

ΔPST and ΔQRT are both triangles in the sketch, hence we will attempt to prove that ΔPST and ΔQRT are similar.

In ΔPST and ΔQRT:

$\hat{P}_2 = \hat{Q}_1$ (∠s in same segment)

$\hat{S}_2 = \hat{R}_1$ (∠s in same segment)

$\hat{T}_1 = \hat{T}_3$ (3^{rd}∠ of Δ)

∴ ΔPST ||| Δ QRT (∠∠∠)

Pay attention to the order of letters

∴ $\frac{\text{PS}}{\text{QR}} = \frac{\text{ST}}{\text{RT}}$ (||| Δs)

b) ΔPQRS??? ✗

$\frac{\text{PQ}}{\text{PT}} = \frac{\text{SR}}{\text{ST}}$

The top sides don't give a triangle in the sketch

ΔPQT ✓ $\left| \frac{\text{PQ}}{\text{PT}} = \frac{\text{SR}}{\text{ST}} \right|$ ΔRST ✓

ΔPQT and ΔRST are both triangles in the sketch, hence we will attempt to prove them similar.

In ΔPQT and ΔRST:

$\hat{P}_3 + \hat{P}_4 = \hat{S}$ (∠s in same segment)

$\hat{Q}_2 = \hat{R}_2$ (∠s in same segment)

$\hat{T}_4 = \hat{T}_2$ (3^{rd}∠ of Δ)

∴ ΔPQT ||| Δ SRT (∠∠∠)

Pay attention to the order of letters

∴ $\frac{\text{PQ}}{\text{PT}} = \frac{\text{SR}}{\text{ST}}$ (||| Δs)

(c)

$PM \cdot \boxed{ST} = \boxed{RS} \cdot MT \rightarrow \frac{PM}{RS} = \frac{MT}{ST}$

ΔMPT ✓

$\frac{PM}{RS} = \frac{MT}{ST}$

ΔRST ✓

Prove that ΔMPT and ΔRST are similar.

In ΔMPT and ΔRST:

$\hat{P}_3 = \hat{R}_2$ (alt ∠s ; PN ∥ SR)

$\hat{M}_1 = \hat{S}_3$ (alt ∠s ; PN ∥ SR)

$\hat{T}_4 = \hat{T}_2$ (3^{rd}∠ of Δ)

$\therefore \Delta PMT \, ||| \, \Delta RST$ (∠∠∠)

$\therefore \frac{PM}{RS} = \frac{MT}{ST}$ (||| Δs)

$\therefore PM \,.\, ST = RS.\,MT$

(d) **Rewrite the square as a product**

$\mathbf{AP^2 = AR.\,AS}$

$\therefore \mathbf{AP\,.\,AP = AR\,.\,AS}$

$AP \cdot \boxed{AP} = \boxed{AR} \cdot AS \rightarrow \frac{AP}{AR} = \frac{AS}{AP}$

ΔAPS ✓

$\frac{AP}{AR} = \frac{AS}{AP}$

ΔAPR ✓

Prove that ΔAPS and ΔAPR are similar.

In ΔAPS and ΔAPR:

$\hat{P}_1 = \hat{R}_2$ (tan chord thm)

$\hat{A} = \hat{A}$ (common)

$\hat{S}_1 = \hat{P}_1 + \hat{P}_2$ (3^{rd}∠ of Δ)

$\therefore \Delta PAS \, ||| \, \Delta RAP$ (∠∠∠)

Pay attention to the order of letters

$\therefore \frac{PA}{RA} = \frac{AS}{AP}$ (||| Δs)

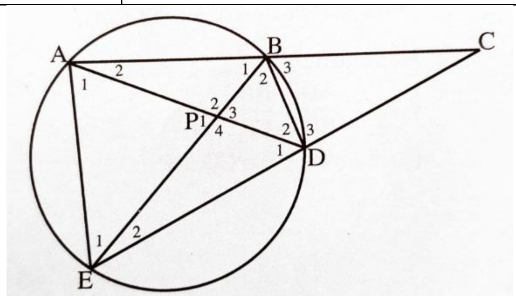
$\therefore AP^2 = AR.\,AS$

CAN YOU:

In the sketch alongside, prove that

(a) $\frac{AB}{ED} = \frac{AP}{EP}$

(b) $AE\,.\,CD = AC\,.\,BD$



Proof of Theorem 2.

If two triangles are equiangular, then their corresponding sides are in the same proportion and hence the triangles are similar.

Given: $\Delta\mathbf{ABC}$ and $\Delta\mathbf{DEF}$ with $\hat{\mathbf{A}} = \hat{\mathbf{D}}$, $\hat{\mathbf{B}} = \hat{\mathbf{E}}$ **and** $\hat{\mathbf{C}} = \hat{\mathbf{F}}$

Required to prove : $\frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{AC}}{\mathbf{DF}} = \frac{\mathbf{BC}}{\mathbf{EF}}$ and hence $\Delta\mathbf{ABC}\,|||\,\Delta\,\mathbf{DEF}$

Proof:

Construction: Mark off P on AB and Q on AC, such that AP = DE and AQ = DF. Draw PQ.

In ΔAPQ and Δ DEF :

1. $\hat{A} = \hat{D}$ (given)
2. $AP = DE$ (construction)
3. AQ = DF (construction)

$\therefore$ $\Delta APQ \equiv \Delta DEF$ (S; ∠; S)

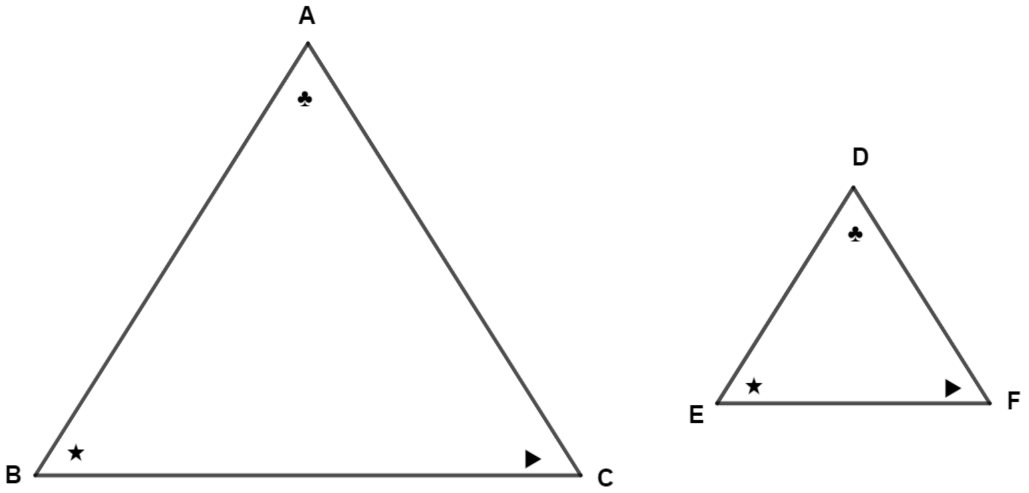
$\therefore A\hat{P}Q = \hat{E}$ (≡ ∆s)

And $\hat{E} = \hat{B}$ (given)

$\therefore A\hat{P}Q = \hat{B}$

$\therefore$ PQ ∥ BC (corresp ∠s =)

$\therefore \frac{AB}{AP} = \frac{AC}{AQ}$ (line ∥ side of ∆)

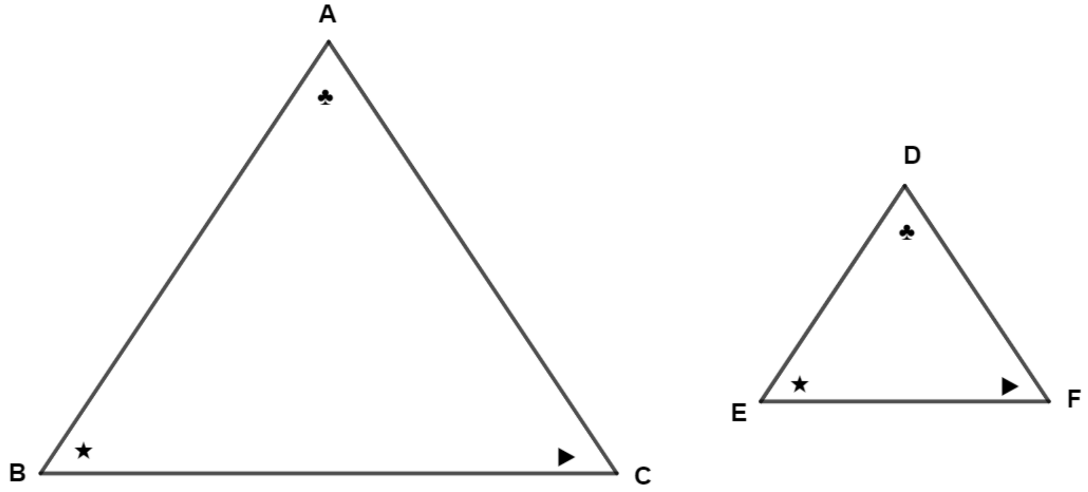


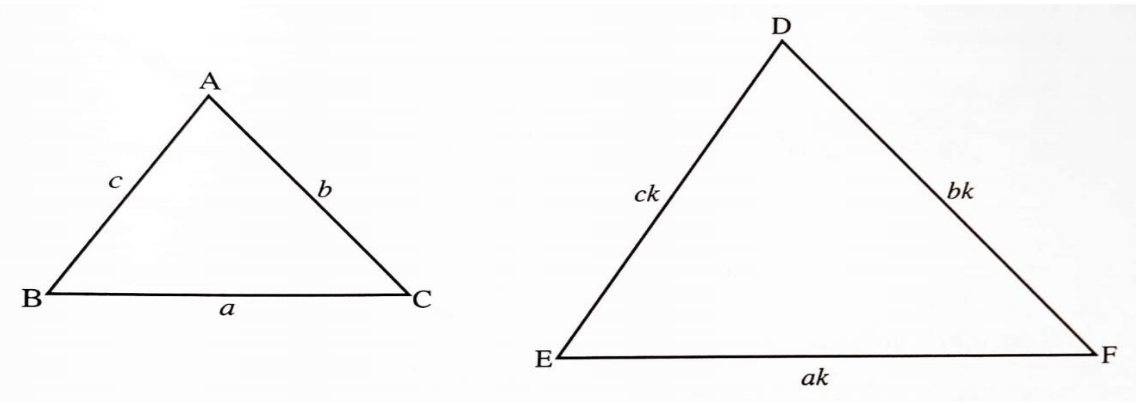
But AP = DE and AQ = DF

$\therefore \frac{AB}{DE} = \frac{AC}{DF}$

Simarlarly, by marking of P on BA and Q on BC, such that BP = ED and BQ = EF, it can be shown that $\therefore \frac{AB}{DE} = \frac{BC}{EF}$

$\therefore \frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{AC}}{\mathbf{DF}} = \frac{\mathbf{BC}}{\mathbf{EF}}$

ACTIVITIES/ ASSESSMENT	<table><tr><th>Clever</th><th>Mind Action Series</th><th>Classroom Mathematics</th><th>Via Afrika Mathematics</th></tr><tr><td>Ex: 11.3 & 11.4 Pg: 288 & 293</td><td>Ex: 5 & 6 Pg: 249 & 256</td><td>Ex: 11.3 Pg: 295</td><td>Ex:4 Pg: 242</td></tr></table>
CONSOLIDATION	**If two triangles are equiangular, then their corresponding sides are in the same proportion and hence the triangles are similar**.  **Given**: $\Delta\mathbf{ABC}$ and $\Delta\mathbf{DEF}$ with $\hat{\mathbf{A}} = \hat{\mathbf{D}}$, $\hat{\mathbf{B}} = \hat{\mathbf{E}}$, $\hat{\mathbf{C}} = \hat{\mathbf{F}}$ **Then:** $\frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{AC}}{\mathbf{DF}} = \frac{\mathbf{BC}}{\mathbf{EF}}$ **and** $\Delta\mathbf{ABC} \mid\mid\mid \Delta\,\mathbf{DEF}$ **.**

	CONVERSE OF THEOREM 2. If the corresponding sides of two triangles are in the same proportion, then the triangles are equiangular and hence similar.  **Given**: $\boldsymbol{\Delta ABC}$ and $\boldsymbol{\Delta DEF}$ with $$\frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{AC}}{\mathbf{DF}} = \frac{\mathbf{BC}}{\mathbf{EF}}$$ **Then:** $\hat{\mathbf{A}} = \hat{\mathbf{D}}$, $\hat{\mathbf{B}} = \hat{\mathbf{E}}$, $\hat{\mathbf{C}} = \hat{\mathbf{F}}$ **and** $\boldsymbol{\Delta ABC \,|||\, \Delta DEF}$.