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| SUBJECT and GRADE | Mathematics Grade 12 |
| TERM 2 | Week 4 |
| TOPIC | Calculus |
| AIMS OF LESSON | • Determine the average gradient between two points of a function.<br>• Define the term derivative.<br>• Determining the derivative from first principles. Determine the derivative apply the differentiation rules.<br>• Determine the equation of the tangent given certain information |

| RESOURCES | Paper based resources | Digital resources |
|---|---|---|
| | *Please go to the Calculus Topic in your Mathematics Textbook.* | https://www.youtube.com/watch?v=kyErKnDwDXM<br>https://www.youtube.com/watch?v=vzDYOHETFlo<br>https://www.youtube.com/watch?v=WHnRyzXXT1U |

**INTRODUCTION:**

Dear learner in earlier grades you have learnt about the **Gradient** between two points on a straight line. You have used the formula, $\boldsymbol{gradient} = \frac{y_2-y_1}{x_2-x_1}$, to determine the gradient between any two points. You also know that on a straight line the gradient at every point on the straight line is the same.

**CONCEPTS AND SKILLS:**

**Average gradient**

Consider a function that is not a straight line like the function, $f(x) = x^2 - 2x - 2$ then the gradient at all points are not the same. So, when we want to find the gradient between any two points on the function, we refer to it as the **average gradient.**

$\boldsymbol{Ave\ gradient\ of\ curve} = \frac{y_2-y_1}{x_2-x_1}$ Using functional notation the two points can be written as: $(x_1; f(x_1))$ and $(x_2; f(x_2))$

$\therefore$ $\boldsymbol{Average\ gradient} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ $\quad \frac{[difference\ in\ y]}{[difference in\ x]}$

**Example 1:**

Determine the average gradient of the graph of $y = 3x^2 - 4$, between $x = -3$ and $x = -1$.

**Solution:**

$$Average\ gradient = \frac{f(-3)-f(-1)}{-3-(-1)} = \frac{23-(-1)}{-2} = \frac{24}{2} = 12$$

**CAN YOU?**

Determine the average gradient of the graph $y = 5x^2 - 4$ between:

1. $x = 1$ and $x = 3$
2. $x = -4$ and $x = -1$
3. $x = 2$ and $x = 3$

Suppose that $x_1 = a$ and $x_2 = a + h$, where $h$ is the distance between the $x$- coordinates of the two points. Then $f(x_1) = f(a)$ and $f(x_2) = f(a+h)$

$\therefore$ $Average\ gradient = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$ , $\therefore$ $Average\ gradient = \frac{f(a+h)-f(a)}{h}$ is average gradient between any points $(a; f(a))$ and $(a+h; f(a+h))$ where $h$ is the distance between the $x$- coordinates of the two points. Thus we can say, $Average\ gradient = \frac{f(x+h)-f(x)}{h}$ .

Right hand side is read as, "the limit of the average gradient as $h$ tends to 0".

**Derivative:** Derivative of a function gives, the slope/gradient of the function **at any point.**

**Definition of Derivative:** Derivative of a function $f(x)$ is defined as, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Left hand side is read as, Derivative of a function $f$

**Note:** lim, is the abbreviated notation for limit.

$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ , this represents the gradient at a point where $x = a$

$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$ , this represents the gradient at a point where $x = 1$

The derivative or the gradient at any point of a function can be determined using **first principles** or using rules.

When asked to find the derivative using **First Principles** the following formula must be used:

$$\boldsymbol{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

**Example 2**

Determine $f'(x)$ from first principles if $f(x) = 2x^2$.

**Solution:**

$f(x) = 2x^2$

$f(x+h) = 2(x+h)^2 = 2(x^2+2xh+h^2) = 2x^2+4xh+2h^2$

$f(x+h) - f(x) = 2x^2+4xh+2h^2-2x^2 = 4xh+2h^2$

Expand and simplify

Substitute $f(x+h) - f(x)$ into the formula

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ — Formula

$f'(x) = \lim_{h\to 0}\frac{4xh+2h^2}{h}$ — Substitute: $f(x+h) - f(x) = 4xh+2h^2$

$f'(x) = \lim_{h\to 0}\frac{h(4x+2h)}{h}$ — We factorise to eliminate $h$ in the denominator

$f'(x) = \lim_{h\to 0} 4x+2h$

$f'(x) = 4x + 2(0)$

$f'(x) = 4x$

Once simplified, substitute $h = 0$.
The limit sign now disappears!

**CAN YOU?**

A. Use first principles to determine the derivative of $f$ if:

1. $f(x) = x^2 + 3$
2. $f(x) = x^3$
3. $f(x) = -2x^2$
4. $f(x) = -3x$
5. $f(x) = \frac{2}{x}$

**Example 3:**

(a) Determine $f'(x)$ if $f(x) = x^2 - 5x$ from first principles.

**Solution:**

$f(x) = x^2 - 5x$

$f(x+h) = (x+h)^2 - 5(x+h) = x^2 + 2xh + h^2 - 5x - 5h$

$f(x+h) - f(x) = \left(x^2 + 2xh + h^2 - 5x - 5h\right) - (x^2 - 5x) = 2xh + h^2 - 5h$

Expand and simplify

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Formula

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2 - 5h}{h}$$

Substitute $f(x+h) - f(x) = 2xh + h^2 - 5h$

$$f'(x) = \lim_{h \to 0} \frac{h(2x + h - 5)}{h}$$

We factorise to eliminate $h$ in the denominator

$$f'(x) = \lim_{h \to 0} 2x + h - 5$$

Once simplified, substitute $h$ as 0. The limit sign now disappears!

$$f'(x) = 2x + 0 - 5$$

$$f'(x) = 2x - 5$$

b) Hence determine the gradient of the tangent at $x = -1$.

$m = f'(x)$ at $x = -1$.

$f'(x) = 2x - 5$

$f'(-1) = 2(-1) - 5$

$f'(-1) = -7$

To find the derivative(gradient) at the point where $x = -1$, substitute the $x$ value.

B. Given the functions defined by the following equations:

- $y = 3x$
- $y = 5$
- $y = -2x$

1. Write down the gradients of each of the functions.
2. Use first principles to find the derivative of each of these functions
3. Compare your answers of 1 and 2. Can you explain what you find?

C. If $g(x) = x^2 - 2$, determine:

1. Average gradient between $x = -3$ and $x = -1$.
2. $g'(x)$ from first principles.
3. $g'$(-2)
4. the gradient of the tangent at $x = -2$.

# Rules for Differentiation

Power rule states that: If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$

We multiply the coefficient $x$ by the exponent and reduce the exponent by 1.

There are different notations for the derivative.

| Notation for derivative | **We differentiate w.r.t $x$** | Example 4 |
|---|---|---|
| $f'(x)$ | Find $f'(x)$ if $f(x) = x^4$ | $f(x) = x^4$<br>$f'(x) = 4x^3$ |
| $y'$ | Find $y'$ if $y = x^{-1}$ | $y = x^{-1}$<br>$y' = -x^{-2}$ |
| $\frac{dy}{dx}$ | Find $\frac{dy}{dx}$ if $y = 2x^3$ | $y = 2x^3$<br>$\frac{dy}{dx} = 6x^2$ |
| $\frac{d}{dx}$ | Find $\frac{d}{dx}[2x]$ | $\frac{d}{dx}[2x^1] = 2x^0 = 2$ |
| $D_x[f(x)]$ | Find $D_x[-2x^{-4}]$ | $D_x[-2x^{-4}] = 8x^{-5}$ |

Multiply the exponent by the coefficient of $x$.

In, $x^4$, the Exponent is 4 and Coefficient is 1

$\therefore 4 \times 1 = 4$

Please note:

**1 less than -4 is -5**
**or $-4 - 1 = -5$**

**Important Notes:**

- $D_x[x] = 1$
- **The derivative of a constant is 0.**

**Can you?**

1. Find $\frac{dy}{dx}$ if $y = 3x^{\frac{3}{4}}$
2. Find $\frac{d}{dx}[4x^{-3}]$
4. Find $y'$ if $y = \frac{x^3}{2}$

We can apply this rule to individual terms of a function.

Example 5

Determine:

$f'(x)$ *if* $f(x) = \sqrt{x^3} - \frac{4}{x^3}$

- Remove surd
- Re-write fraction

**Solution:**

$f(x) = x^{\frac{3}{2}} - 4x^{-3}$

Apply rule to each term

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} + 12x^{-4}$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} + \frac{12}{x^4}$

Answer with positive exponent

Example 6: Determine:

$D_x\left[\left(x^2 - \frac{1}{x^2}\right)^2\right]$

**Solution:**

Remove brackets and simplify

$D_x\left[\left(x^2 - \frac{1}{x^2}\right)^2\right]$

$= D_x\left[\left(x^4 + 2(x^2)\left(\frac{1}{x^2}\right) + \frac{1}{x^4}\right)\right]$

$= D_x[x^4 + 2 + x^{-4}]$

$= 4x^3 + 0 - 4x^{-5}$

$= 4x^3 - \frac{4}{x^5}$

Write each term so that there is no $x$ in the denominator

Answer with positive exponent

**Before applying the rule make sure that:**

- Remove all brackets
- Simplifying fractions by factorizing or cancelling
- Write expression without surd
- Write each term so that there is no $x$ in the denominator: $\frac{1}{x} = x^{-1}$

**Can you?**

1. Determine:

(b) $D_x[\sqrt{x} - \frac{1}{2}]$

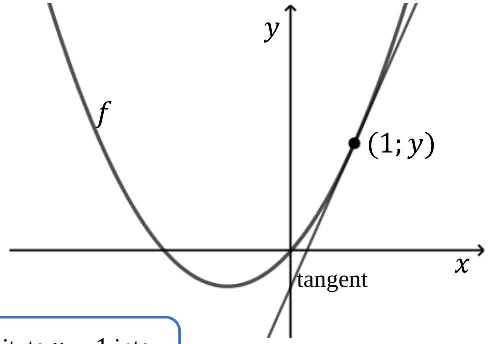
2. Determine $f'(x)$ in the following

(a) $f(x) = (3x + 2)(x - 4)$

(b) $f(x) = \left(x - \frac{1}{2}\right)^2$

**The equation of the tangent at a point on a function**

Now it is easy to find the equation of the tangent




**Example 7**

**If $y = x^2 + 2x$, determine the equation of tangent at $x = 1$.**

**Solution:**

$\frac{dy}{dx} = 2x + 2$

$m_{tangent} = 2(1) + 2 = 2 + 2 = 4$

Substitute $x = 1$ into derivative

We need the coordinates of the point

$\therefore y = (1)^2 + 2(1) = 3$  Point $(1; 3)$

Substitute $x = 1$ into the function

Equation of tangent: $y - y_1 = m(x - x_1)$

Equation of tangent: $y - 3 = 4(x - 1)$

$y = 4x - 4 + 3$

$y = 4x - 1$

Substitute $m$ and point in

$y - y_1 = m(x - x_1)$

**Example 8**

**Find the equation of the tangent to the function $f(x) = x^3 + 2x + 4$**

**At the point where $x = -1$**

**Solution:**

$f(x) = x^3 + 2x + 4$

$f'(x) = 3x^2 + 2$

Step 1: Derivative

$f'(-1) = 3(-1)^2 + 2$

$= 3 + 2 = 5$

$m = 5$

Step 2: $f'(x)$ , is gradient at any point
$f'(-1)$, is the gradient at $x = -1$

Coordinates of the point:

$f(-1) = -1 - 2 + 4 = 1$

Coordinates $(-1; 1)$

Step 3: Coordinates of point:
Substitute into $f(x)$

**Important Notes**

- A tangent is a line which touches the function at one point.
- The gradient of the tangent at the point of contact is the derivative.
- Equation of a line is:

$$y = mx + c$$

or

$$y - y_1 = m(x - x_1).$$

Equation of tangent:

$y - y_1 = m(x - x_1)$

$y - 1 = 5(x + 1)$

$y = 5x + 5 + 1$

$y = 5x + 6$

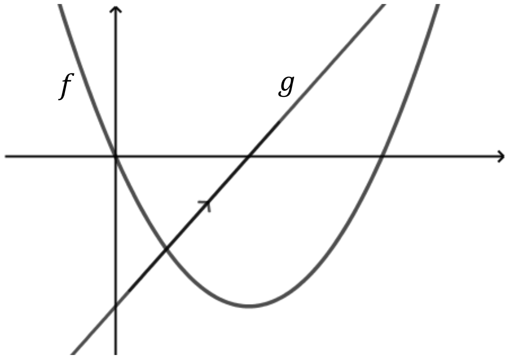
Step 4: Substitute $m$ and point into $y - y_1 = m(x - x_1)$

The equation of the tangent at $x = -1$ is:

$y = 5x + 6$

**Example 9**

**Find the equation of the tangent to $f(x) = x^2 - 4x$, parallel to $g(x) = 2x - 4$**




**Solution:**

$f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

Tangent is parallel to $g(x)$

$\therefore 2x - 4 = 2$

$2x = 6$

$x = 3$

// lines

Gradients equal

Point of contact

The point of contact is therefore $(3; y)$

$f(3) = (3)^2 - 4(3) = 9 - 12 = -3$

Point $(3; -3)$

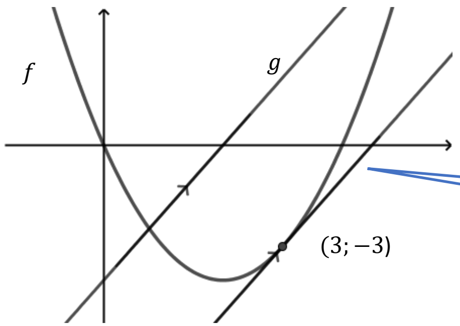
Equation of tangent:

$y - y_1 = m(x - x_1)$

$y + 3 = 2(x - 3)$

$y = 2x - 6 - 3$

$y = 2x - 9$




tangent

**Can you?**

1. Determine the equation of the tangent to:
   (a) $y = 2\sqrt{x}$ at $x = 9$
   (b) $y = \dfrac{9}{x}$ at $x = 1$
   (c) $y = \dfrac{4x^2 - 1}{2x - 1}$ at $x = 2$
   (d) $f(x) = \dfrac{x^3 - 2x^2 - 3}{x}$ at $x = -1$
2. Determine the equation of the tangent to the curve $f(x) = -x^2 + 3x$ which is parallel to the line $y = x + 2$
3. Find the value of $p$ if $y = p - 9x$ is the tangent to $y = -x^3 + 3x - 2$.
4. Find the value of $t$ if the line $y + 2x = t$ is the tangent to $f(x) = 5 + 4x - x^2$.
5. The graph of $f(x) = x^2 + 6$ has a tangent at $x = a$. The tangent paases through the point $(2; 1)$. Determine the equation of the tangent.

**Example 10:**

**Determine the equation of the tangent at the turning point of $f(x) = -2(x-1)^2 + 3$.**

**Solution:**

$f(x) = -2(x^2 - 2x + 1) + 3$

$f(x) = -2x^2 + 4x - 2 + 3$

$f(x) = -2x^2 + 4x + 1$

$f'(x) = -4x + 4$

Point of contact is the turning point with coordinates $(1;3)$

$\therefore f'(1) = -4(1) + 4$

$f'(x) = 0$

Equation of the tangent:

$y - y_1 = m(x - x_1)$

$y - 3 = 0(x - 1)$

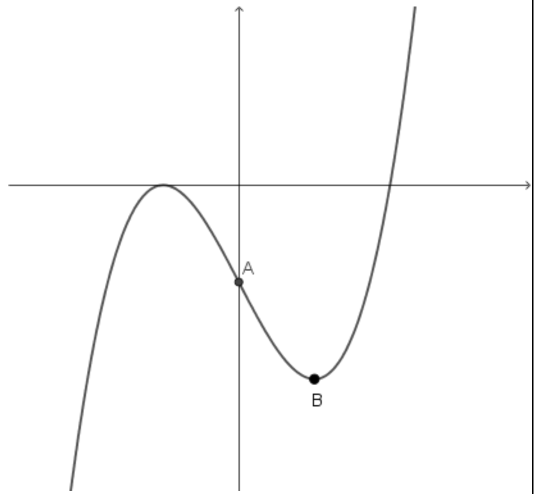
$y - 3 = 0$

$y = 3$

Please note that the gradient of the tangent at the turning point of a function is 0.

In other words, **the derivative at the turning point of a function is 0.**

6. Sketched below is the graph of , $f(x) = x^3 - 3x - 2$




(a) Write down the coordinates of A
(b) Determine the derivative at A.
(c) Write down the derivative at B.
(d) Hence, determine the coordinates of B.

ACTIVITIES/ ASSESSMENT

| Topic | Mind Action Series | Platinum | Classroom Mathematics | Everything Mathematics |
|---|---|---|---|---|
| Ave gradient | Ex: 2; Pg: 166 | Ex: 2&3; Pg: 137; 140 | Ex: 8.1; Pg: 189 | Ex: 6.2; Pg: 218 |
| First principles | Ex: 3;Pg: 171 | Ex: 4 & 5 ; Pg: 145 | Ex: 8.4; Pg: 199 | Ex: 6.3; Pg: 220 |
| Rules of differentiation | Ex 4; Pg 177 | Ex 6; Pg 179 | Ex 8.6; Pg 204 | Ex: 6 – 4 Pg 223 |
| Equations of tangents | Ex 9; Pg 197 | Ex 8; Pg 153 | Ex 8.7; Pg 207 | Ex: 6 – 5 Pg 228 |

CONSOLIDATION

- Deriving the derivative from first principles is very important. The formula is on the formula sheet.
- The **derivative** of a function is the **gradient** of the function at any point.
- The derivative of a constant is 0. $\pi$ is a constant therefore $D_x[\pi] = 0$
- We need $m$ and a point to determine the equation of a line.
- The derivative is gradient $(m)$ of the tangent of a function at any point.
- The derivative at a turning point is 0.