SUBJECT and GRADE	MATHEMATICS GR 12
TERM 2	Week 7
TOPIC	INVERSE FUNCTIONS
AIMS OF LESSON	To:

- Understand the definition of a function.
- Learn about the inverse of prescribed functions: $y = ax + q$; $y = ax^2$
- Learn how to sketch and interpret the graphs of inverse functions.

RESOURCES	***Paper based resources***	***Digital resources***
	Please go to the Chapter on Inverse Functions in your Mathematics Textbook.	https://youtu.be/hXb2FEv8Mso- https://video.tutonic.org/T12inversefunction

INTRODUCTION:

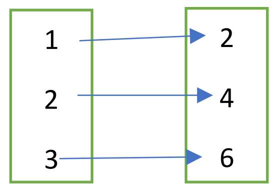
- In grades 10 and 11, you learnt how to draw and interpret the graphs of different functions, namely: the straight line: $y = ax + q$; the parabola: $y = a(x-p)^2 + q$; the hyperbola: $y = \frac{a}{x-p} + q$ and the exponential function: $y = ab^{x-p} + q$.
- You also identified the following for the above functions: domain, range, intercepts with the axes, turning points, minima, maxima, asymptotes, shape, symmetry and the intervals on which the function increases/decreases.
- And generalised the effects of a, k, p and q on the graphs of the above functions.
- We will now build on this knowledge to study inverse functions

CONCEPTS AND SKILLS

Definition of a function

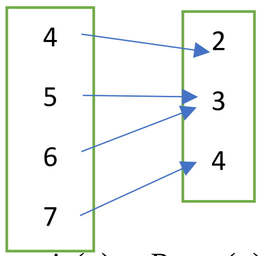
A relation is any relationship between two variables. A function is a special kind of relation in which: For every x-value, there is at most one y-value. Each element of the domain (x) is associated with only one element of the range(y). In other words, the x-values are never repeated in the set of ordered pairs of a function.
If each element of the domain is associated with only one element of the range. The relation is a **one-to-one function**.

Example 1: {(1; 2); (2; 4); (3; 6)} is a function



Domain(x) Range(y)

Example 2:
{ (4; 2); (5; 3); (6; 3); (7; 4)} the relation is said to be a **many-to-one function**. Each x-value still associates with only one y-value.



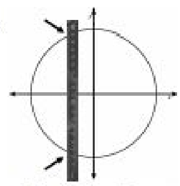
Domain(x) Range(y)

HOW TO DETERMINE WHETHER A GRAPH IS A FUNCTION:

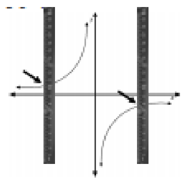
The Vertical and Horizontal Line Tests

You can use a ruler to perform the "**vertical line test**" on a graph to see whether it is a function or not. Hold a clear plastic ruler parallel to the y-axis, i.e. vertical. Move it from left to right over the axes. If the ruler only ever cuts the curve in **one** place, then the graph **is** a function. If the ruler at any stage cuts the graph in more than one place, then the graph is not a function. This is because the same x-value will be associated with more than one y-value.

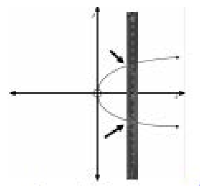
The "**horizontal line test**" determines if the graph is a one-to-one or many-to-one function. If the ruler is positioned horizontally so that it is parallel to the x-axis, and the movement of the ruler is horizontally up or down, the following holds true: If the ruler only ever cuts the curve in one place, then the graph is a one-to-one function. If the ruler at any stage cuts the graph in more than one place, then the graph is a many-to-one function. See the graphs below.

1.


Not a function

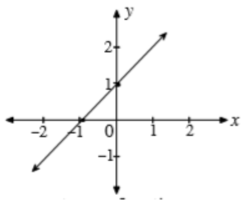
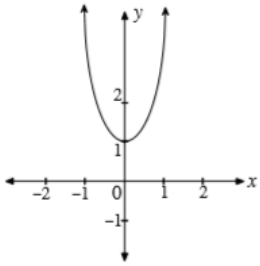
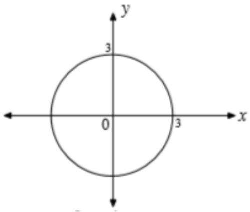
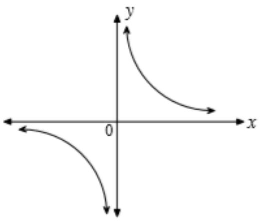
2.


function

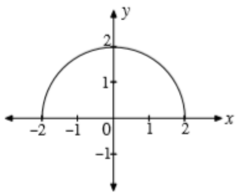
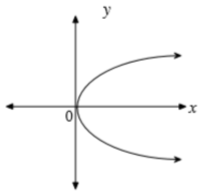
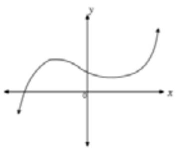
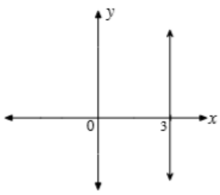
3


Not a function

Example 3:
Determine whether the following relations are functions or not. If the graph is a function, determine whether the function is one-to-one or many-to-one.

1. one- to- one function 	2. one- to-many function 
3. Not a function 	4. one- to- one function 

Can you:
Identify whether the following relations are functions or not. If the graph is a function, determine whether it is one-to-one or many-to-one.

1. 	2. 
3. 	4. 

Answer: 1. Many-to-one function 2. Not a function
3. many-to-one 4. Not a function

Functional notation:
Since functions are special relations, we reserve certain notation strictly for use when dealing with functions. Consider the function $f = \{(x; y) \,/ y = 3x\}$ This function may be represented by means functional notation.

Functional notation
$f(x) = 3x$ This is read as "f of x is equal to $3x$". The symbol $f(x)$is used to denote the element of the range to which x maps. In other words, the y-values corresponding to the x-values are given by $f(x)$, i.e. $y = f(x)$.
For example, if $x = 4$, then the corresponding y-value is obtained by substituting $x = 4$ into $3x$. For $x = 4$ the y-value is $f(4) = 3(4) = 12$. The brackets in the symbol $f(4)$ do not mean f times 4.

INVERSES OF ONE-TO-ONE LINEAR FUNCTIONS

An inverse function of a function f, is a function which does the "reverse" of a given function f.
f^{-1}, is the notation used for the inverse of function f .

Consider the function $f(x) = 2x - 1$
f is the rule that maps values in the domain (x) to values in the range (y). Note: $2x - 1 = y$.
If $x = 3$, then the function f maps this x-value to a corresponding y- value in the range as follows:

$$f(x) = 2x - 1$$
$$\therefore f(3) = 2(3) - 1$$
$$\therefore f(3) = 5$$
$$\therefore y = 5$$

So if $x = 3$, then $y = 5$
The rule that reverses this process and maps 5 back to 3 is called the inverse of the original function f and is denoted by f^{-1}.

<table>
<tr><th>Method 1: Inverse function by swopping x and y</th><th>Method 2: Inverse function using flow diagrams</th></tr>
<tr><td>So if $y = 2x - 1$ (f)
Then $x = 2y - 1$ (interchange x and y)
$\therefore -2y = -x - 1$
$\therefore 2y = x + 1$
$\therefore y = \frac{x+1}{2}$
We then say, inverse function of f is:
$f^{-1}(x) = \frac{x+1}{2}$

If $x = 5$, is substituted into f^{-1}, then
$y = \frac{5+1}{2} = 3$
So rule f, maps 3 onto 5 and,
the reverse (or inverse rule)
f^{-1} maps 5 back onto 3.</td><td>A flow diagram could also help you to understand the concept of inverse functions:

If $f(x) = 2x - 1$, complete a flow diagram the function

$x \rightarrow$ [multiply by 2] $\rightarrow 2x \rightarrow$ [subtract 1] $\rightarrow 2x - 1$

So the inverse does the reverse, so you perform the operations back to front. That which was performed last is now performed first.

Inverse: $x \rightarrow$ [Add 1] $\rightarrow x + 1 \rightarrow$ [Divide by 2] $\rightarrow \frac{x+1}{2}$

The inverse function as of f is, $f^{-1}(x) = \frac{x+1}{2}$</td></tr>
<tr><td>Can You:
Determine the inverse function of,
$f(x) = -3x + 4$
by:
1. Method 1
2. Method 2 above.</td><td>Solution: $f^{-1}(x) = \frac{x-4}{-3}$</td></tr>
</table>

Given the function $f(x)$, we determine the inverse $f^{-1}(x)$ by:

- Interchanging x and y in an equation;
- Making y the subject of the equation;
- Expressing the new equation in function notation.

Example 4:

If $f(x) = 2x - 4$

1. Determine f^{-1}, that is the inverse of f.
2. Sketch the graphs of f , f^{-1} and $y = x$ on the same system of axes.
3. Determine the coordinates of the point of intersection and indicate it on your sketch.
4. Write down the domain and range for f and f^{-1}.

Solution:

1.

So if, $f(x) = 2x - 4$

Then: $y = 2x - 4$ (f)

Then: $x = 2y - 4$ (interchange x and y)

$\therefore -2y = -x - 4$

$\therefore 2y = x + 4$

$\therefore y = \frac{x+4}{2}$

We then say, inverse function of f is:

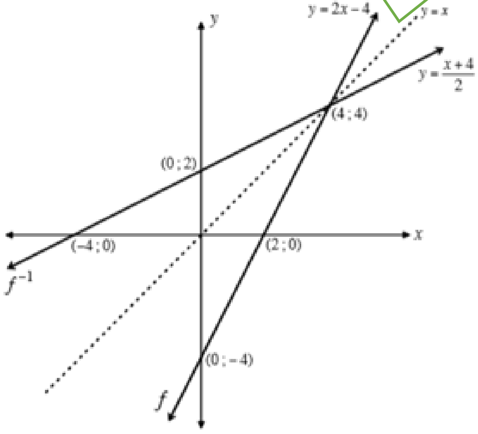
$f^{-1}(x) = \frac{x+4}{2}$

2.
Sketching of these functions:
$y = 2x - 4 \quad (f)$
$\boldsymbol{y}$ **– intercept: let** $\boldsymbol{x = 0}$
$y = 2(0) - 4 = -4$
$\boldsymbol{x}$ **– intercept: let** $\boldsymbol{y = 0}$
$0 = 2x - 4$
$\therefore -2x = -4$
$\therefore x = 2$

Only the inverse of a one-to-one function is itself a function.

$y = \frac{x+4}{2} \quad (f^{-1})$
$\boldsymbol{y}$ **– intercept: let** $\boldsymbol{x = 0}$
$y = \frac{0+4}{2} = 2$
$\boldsymbol{x}$ **intercept: let** $\boldsymbol{y = 0}$
$0 = \frac{x+4}{2}$
$\therefore x = -4$

The inverse function is symmetrical to the function, about the line $y = x$



3. You can also find the point of intersection of these two graphs by solving the equation:

$f(x) = f^{-1}(x)$
$\therefore 2x - 4 = \frac{x+4}{2}$
$\therefore 4x - 8 = x + 4 \quad$ (LCD = 2)
$\therefore 3x = 12$
$\therefore \boldsymbol{x = 4}$
$\therefore y = 2(4) - 4$
$\therefore \boldsymbol{y = 4}$
The coordinates of the point of intersection are (4; 4)

4.
Domain of f: $x \in R$
Range of f: $y \in R$
Domain of f^{-1}: $x \in R$
Range of f^{-1}: $y \in R$

Do you notice that:
The function $f(x) = 2x - 4$ is a one-to-one linear function.
And it's inverse $f^{-1}(x) = \frac{x+4}{2}$ is also a one-to-one function

CAN YOU
If $f(x) = -3x + 6$
1. Write the equation of the inverse in the form f^{-1}
2. Sketch the graphs of f and f^{-1} on the same system of axes, along with the line $y = x$.
3. Find the point of intersection and indicate it on your sketch.

Answer: 1. $f^{-1}(x) = \frac{x-6}{-3}$ 3. $(\frac{6}{5}; 3\frac{3}{5})$

INVERSES OF MANY- TO- ONE QUDRATIC FUNCTION

Consider the many –to-one function $\boldsymbol{f(x) = x^2}$

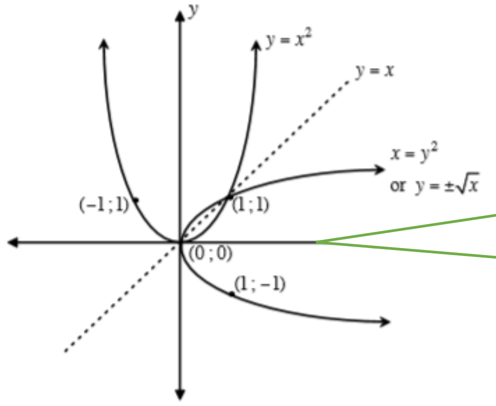
Lets sketch: $\boldsymbol{f(x) = x^2}$ **which can also be written as** $y = x^2$, by using the table with some values for x.

x	-1	0	1
y	1	0	1

Interchange x and y of, $y = x^2$, then $x = y^2$
Use the table below to sketch, $x = y^2$

y	-1	0	1
x	1	0	1

It is possible to make y the subject of the formula for the inverse relation.
$\therefore y^2 = x$
$\therefore y = \pm\sqrt{x}$ **Provided** $x \geq 0$
The graph of $y = \pm\sqrt{x}$ is not a function because a vertical line will cut the graph in two points as it moves from left to right. So we will need to do something to the graph of $y = x^2$ so that when we determine the inverse, this inverse will also be a function



The inverse of a parabola is not a function, because there are elements of the domain which have more than one y-value associated with it.

BUT if we **restrict** the domain of the original function, we can get an inverse that is a function

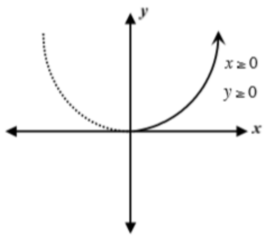
Please note

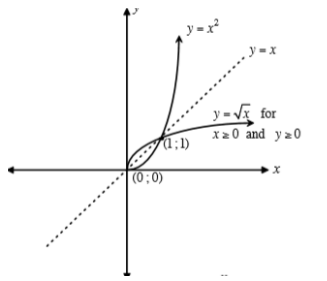
If a function is not a one –to-one function the inverse will not be a function.
However, below you will see that the domain of a many-to-one function can be restricted so that its inverse is a function.
There are two different restrictions that can be placed on the domain so that the inverse is a function.

Method 1

Restrict the domain of $f(x) = x^2$ as follows:
$f(x) = x^2$ *where* $x \geq 0$
Note that the graph of this parabola will be the one half of the parabola, where the x-values are positive.
The range of this function is the same as for the original function, $y \in [0\,;\,\infty)$.
The inverse of the graph of the function f, is the image when f is reflected about $y = x$. See the adjacent sketch.
The equation of the inverse function is then defined as,
$f^{-1}(x) = \sqrt{x}$ where $x \geq 0$ and $y \geq 0$.

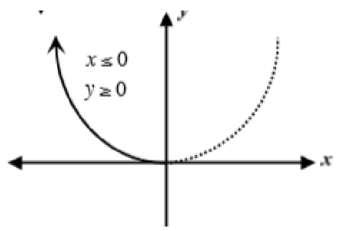
Note, both f and f^{-1} are one-to-one functions.

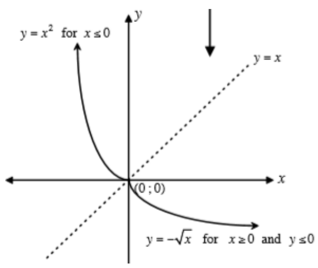




Method 2

Restrict the domain of
$f(x) = x^2$ as follows:
$f(x) = x^2$ where $x \leq 0$
Note that the graph of this parabola will be the half of the parabola, where the x-values are negative.
The range of this function is the same as for the original function, $y \in [0\,;\,\infty)$.
The inverse of the graph of the function f, is the image when f is reflected about $y = x$. See the adjacent sketch.
It is clear that the inverse of the graph of the function





$f(x) = x^2$ where $x \leq 0$ is also a function. The equation of the inverse function is then defined as $f^{-1}(x) = -\sqrt{x}$ where $x \geq 0$ *and* $y \leq 0$
Note, both f and f^{-1} are one-to-one functions.

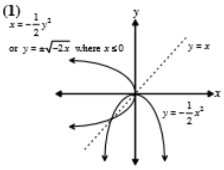
CAN YOU:

Given: $f(x) = -\frac{1}{2}x^2$

1. Sketch the graph of the function and its inverse on the same set of axes.
2. Now restrict the domain of the original function in two different ways so as to form new one-to-one functions.
3. Sketch the graphs of the new function and its inverse function on the same set of axes.
4. Hence rewrite the equation of each inverse function in the form $f^{-1}(x) =$
5. Write down the domain and range for each graph drawn.

Answer:

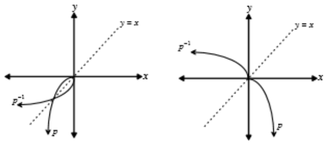
(1)



(2) Situation 1: $p(x) = -\frac{1}{2}x^2$ where $x \leq 0$

Situation 2: $p(x) = -\frac{1}{2}x^2$ where $x \geq 0$

(3) Situation 1   Situation 2



(4) Situation 1: $p^{-1}(x) = -\sqrt{-2x}$ where $x \leq 0$ Situation 2: $p^{-1}(x) = -\sqrt{-2x}$ where $x \leq 0$

(5)

Situation 1		Situation 2	
Domain of p	$x \in (-\infty ; 0]$	Domain of p:	$x \in [0 ; \infty)$
Range of p:	$y \in (-\infty ; 0]$	Range of p:	$y \in (-\infty ; 0]$
Domain of p^{-1}:	$x \in (-\infty ; 0]$	Domain of p^{-1}:	$x \in (-\infty ; 0]$
Range of p^{-1}:	$x \in (-\infty ; 0]$	Range of p^{-1}:	$y \in [0 ; \infty)$

EXERCISES: Inverse Functions
Please do exercises on Inverse functions from your Mathematics textbook.

Siyavula: ex 2-3 pg 61 ; ex 2 – 4 pg 66 ; ex 2-5 pg 68

Mind Action Series Grade 12: ex 2 pg 44 ; ex 3 no 2 pg 45

Classroom Mathematics grade 12 : ex 2.4 pg 52 ; ex 2.5 pg 55

Platinum Math: ex 2 pg 40

CONSOLIDATION:
You have to be able to denote the difference in notation between the function and its inverse. The standard notation is $f(x)$ for the function and $f^{-1}(x)$ for the inverse. Do not confuse $f^{-1}(x)$ with $\frac{1}{f(x)}$ (reciprocal) as they do not represent the same thing.
If you represent a function f and the inverse function f^{-1} graphically, the two graphs are reflected about the line $y = x$.
Any point on the line $y = x$ has x and y coordinates with the same numerical value, therefore interchanging the y and x values makes no difference.
Also note that only one-to-one functions have an inverse function. If the function **is not** one-to-one, the domain of the function must be restricted so that the graph is one-to one. The inverse of the function on the restricted domain will then be a function. The domain of the function is the range of the inverse. The range of the function is the domain of the inverse.