SUBJECT and GRADE	Mathematics GR 12	
TERM 3	Week 1	
TOPIC	Finance	
AIMS OF LESSON	Present Value Annuities	
RESOURCES	***Paper based resources***	***Digital resources***
	Textbook chapter on Finance	https://www.youtube.com/watch?v=tv1qKi7BEP4 https://www.youtube.com/watch?v=xWrAg5A7n4k

INTRODUCTION: Dear Grade 12 learner: Make sure you have worked through the previous lesson where we discussed annuities, which is a series of equal payments at fixed intervals. More specifically, **future value annuities** is an annuity in which the aim of the payment is to **save for the future**. The future value of the annuity will be the total amount **saved.**

In this lesson we will continue to learn about annuities. **Present value annuities are loans**. It is common to work with loans in our lives. Most people cannot afford to pay for a car or a house and will need to take a loan from a bank or financial institution.

Present value annuity is an annuity in which the aim of the payments is to pay back the loan. The **present value** of the annuity is the **total amount borrowed**.

We can derive a formula to determine the total loan available in a similar way to that in which we derived the future value annuity formula.

The present value annuity formula is based on the same three assumptions as the future value annuity formula was based:

- The first payment is made one time period from the present.
- The final payment is the n^{th} payment.
- The regularity of compounding the interest is the same time period as the regularity of the payments made.

Present Value Annuity Formula:

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

- P_v — **Present Value**
- x — **Regular payment**
- n — **Number of payments**
- i — **Interest rate/period**

CONCEPTS AND SKILLS:

Example 1.

David applies for a home loan. The bank charges 11,5% p.a. interest, compounded monthly. David can afford to pay R5 000 per month. The bank offers a loan over 20 years. The first payment will be made one month after the loan is granted. Calculate the amount that David can afford to the nearest rand.

Solution:

$x = R\ 5000\ ; \quad i = \frac{0,115}{12} \quad ;\ n = 20 \times 12 = 240$

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$P = \frac{5000\left[1-\left(1+\frac{0,115}{12}\right)^{-240}\right]}{\frac{0,115}{12}}$$

Note that i is,

- $11{,}5\% = \frac{11{,}5}{100} = 0.115$
- Then $i = \frac{0.115}{12}$, because 11,5 is the rate per year so to determine the monthly rate you divide by 12.

$P = 468\ 854{,}19$

$P = R\ 468\ 854$

Example 2.

Dave buys a car and needs to take out a loan for R130 000. The bank charges. 15,5% p.a. interest, compounded monthly and the loan period will be 4 years.

i) Calculate Dave's monthly payment.

ii) Calculate the total interest that Dave will pay.

Solution:

$P_v = R\ 130\ 000\ ; \quad i = \frac{0,155}{12} \quad ;\ n = 4 \times 12 = 48$

i)

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$130\ 000 = \frac{x\left[1-\left(1+\frac{0,155}{12}\right)^{-48}\right]}{\frac{0,155}{12}}$$

$$x = \frac{130000 \times \frac{0,155}{12}}{\left[1-\left(1+\frac{0,155}{12}\right)^{-48}\right]}$$

$$x = R3\ 651{,}03$$

ii) Total repayments $3\ 651{,}03 \times 48 = R175\ 249{,}44$

Interest = Total Repayments – Loan value

= R175 249,44 – R130 000

= R45 249,44

Deferred Payments: Payments that are not started when the loan is granted. When a car dealer advertises, "Buy now and make your first payment in six months", is the car dealer generous? Do you think you will owe the same amount six months later?

When payments are not started immediately, the amount loaned will become larger, as interest will be added for the time that payments were not made.

Example 3.

On the 1st of February, a student takes out a loan to fund his studies. He agrees to start paying the loan back with equal monthly instalments for 4 years on the 28th of February the following year. The loan amount is R44 000 and the interest will be charged at 13% p.a. interest, compounded monthly. Calculate the monthly payments.

Solution:

$A = P(1+i)^n$

$A = 44\ 000\left(1+\frac{0,13}{12}\right)^{12}$

$A = R50\ 073{,}43$

The first payment is not going to be paid at the end of the first month. The loan will grow for 12 months to a new amount owing. The new amount will be the Present value in the formula.

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$50\ 073{,}43 = \frac{x\left[1-\left(1+\frac{0{,}13}{12}\right)^{-48}\right]}{\frac{0{,}13}{12}}$$

$$x = \frac{R50\ 073{,}43 \times \frac{0{,}13}{12}}{\left[1-\left(1+\frac{0{,}13}{12}\right)^{-48}\right]}$$

$x = R1\ 343{,}34$ per month

This is a loan therefore the Present Value Formula

<u>Outstanding Balance on Loans</u>: The outstanding balance on a loan, at a given moment, is the amount that has to be paid to settle the loan. The outstanding balance is given by the following formula:

Outstanding Balance = Loans with interest to date (Using Compound Interest) − Repayments with interest to date (Using Future Value Formula)

For a normal loan, where the first payment is made one interval after the loan was received, and no payments missed, the outstanding balance immediately after the m −th payment is given by:

$$OB = P(1+i)^m - \frac{x[(1+i)^m - 1]}{i}$$

Example 4.

A loan of R60 000 is paid off over a period of 6 years by equal monthly instalments at an interest rate of 9,5% p.a. compounded monthly. Determine the balance outstanding after 4 years.

Solution: $$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$60\ 000 = \frac{x\left[1-\left(1+\frac{0{,}095}{12}\right)^{-72}\right]}{\frac{0{,}095}{12}}$$

$x = R1\ 096{,}48$ per month

We first need to calculate the monthly repayments so that we will be able to establish how much of the loan has been paid back in the 3 years.

$$OB = P(1+i)^m - \frac{x[(1+i)^m - 1]}{i}$$

$$OB = 60\ 000\left(1+\frac{0{,}095}{12}\right)^{48} - \frac{1096{,}48\left[\left(1+\frac{0{,}095}{12}\right)^{48} - 1\right]}{\frac{0{,}095}{12}} = R23\ 881{,}03$$

CAN YOU?

1) Jake takes out a loan of R3 000 000 from the bank to start his own business. The loan will be paid back over 15 years and the monthly repayments will start one month after the loan is granted. The interest rate is 9% p.a. compounded monthly. Determine:
   a) the value of the monthly payments (to the nearest Rand)
   b) the balance of the loan at the end of 5 years.

2) Gary pays off his R2 000 000 home loan, with monthly payments, over a period of 20 years. The interest on the home loan is 10,8% p.a. compounded monthly.
   a) What is the monthly payment?
   b) What is the outstanding balance on his loan after 10 years?

Answers:

1a) R30 428

1b) R2 402 037,83

2a) R 20 3752,24

2b) R1 491 157,68

Finding the time period:

Remember the Logarithmic laws: **If $x = a^y$ then $y = \log_a x$**

$$2^{x-1} = 12$$
$$x - 1 = \log_2 12$$
$$x = \log_2 12 + 1$$
$$x = 4{,}585$$

$$2^{x-1} = 12$$
$$\log 2^{x-1} = \log 12$$
$$(x-1)\log 2 = \log 12$$
$$x - 1 = \frac{\log 12}{\log 2}$$
$$x = \frac{\log 12}{\log 2} + 1$$
$$x = 4{,}585$$

Add logs on both sides of the equation and apply log laws

Example 5. John has to repay a loan of R375 000. The interest rate is 12% p.a. compounded monthly. He pays back R7 500 per month.

(a) How many payments will John have to make? (b) What will his last payment be?

a)

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$375\ 000 = \frac{7500\left[1-\left(1+\frac{0{,}12}{12}\right)^{-n}\right]}{\frac{0{,}14}{12}}$$

$$\frac{375\ 000\times\frac{0{,}12}{12}}{7\ 500} = 1-\left(1+\frac{0{,}12}{12}\right)^{-n}$$

$$\left(1+\frac{0{,}12}{12}\right)^{-n} = 1-\frac{375\ 000\times\frac{0{,}12}{12}}{7\ 500}$$

$$\left(\frac{101}{100}\right)^{-n} = \frac{1}{2}$$

b) Calculate the outstanding balance after the last full payment (the 69th payment)

$$OB = P(1+i)^m - \frac{x[(1+i)^m - 1]}{i}$$

$$OB = 375\ 000\left(1+\frac{0{,}12}{12}\right)^{69} - \frac{7\ 500\left[\left(1+\frac{0{,}12}{12}\right)^{69}-1\right]}{\frac{0{,}12}{12}}$$

$$OB = R4\ 914{,}59$$

$$\text{Last payment} = 4\ 914{,}59\times\left(1+\frac{0{,}12}{12}\right)$$
$$= R\ 4\ 963{,}74$$

$$-n = \log_{\frac{101}{100}}\left(\frac{1}{2}\right)$$

$$n = -\log_{\frac{101}{100}}\left(\frac{1}{2}\right)$$

$n = 69{,}661.$ He will make 70 payments. (69 payments of R7 500 and 1 smaller payment)

Sinking Funds: When equipment is bought, provision has to be made to replace the equipment in the future. This is done by means of a sinking fund or savings a ccount. To set up a sinking fund, the following has to be calculated:

- What the old equipment will be sold for in the future. (Depreciation)
- What new equipment will cost in the future. (Inflation)
- The difference between these two values (Inflation – Depreciation) is the **Future value** that has to be saved up by the time the equipment has to be replaced.

Example 6.

A company purchases a tractor for R1000 000. The value of the tractor depreciates at a rate of 10% p.a. on the reducing balance. The company wants to buy a new tractor in 10 years' time. Inflation is estimated at 7% p.a. The old tractor will be sold at scrap value after 10 years. To purchase a new tractor, the money obtained from selling the old tractor will be used. A sinking fund is set up to finance the balance. The interest rate for the fund is 9% p.a. compounded monthly. The first payment into the fund is made immediately and the last payment at the end of the 10 years. How much must the company pay into the fund per month?

Solution:

Depreciation: $A = P(1-i)^n$
$= 1000\,000(1-0{,}1)^{10}$
$= R\ 348\ 678{,}44$

Inflation: $A = P(1+i)^n$
$= 1\ 0000\ 000(1+0{,}07)^{10}$
$= R\ 1\ 967\ 151{,}36$

Future Value / Difference: R 1 967151 – R 348 678,44 = R 1 618 472,92

Monthly Payment: $F = \frac{x[(1+i)^n - 1]}{i}$

$$1\ 618\ 472{,}92 = \frac{x\left[\left(1+\frac{0{,}09}{12}\right)^{121} - 1\right]}{\frac{0{,}09}{12}}$$

$$x = \frac{1\ 618472{,}92 \times \frac{0{,}09}{12}}{\left[\left(1+\frac{0{,}09}{12}\right)^{121} - 1\right]}$$

$$x = R\ 8\ 258{,}96$$

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CAN YOU?

1) Jason has to pay off a loan of R75 000. He can afford to pay R1 500 per month. The interest rate is 16,2% p.a. compounded monthly. How many payments will he have to make?
2) A Printing machine is bought for R120 000. The value of the equipment depreciates at 15% p.a. on the reducing balance. The inflation rate is 9% p.a. In 5 years' time the old printing machine will be sold at scrap value. The proceeds of the sale, together with money saved up in a sinking fund, will be used to purchase replacement printing machines.
   a) Calculate the scrap value of the old printing machine after 5 years.
   b) What will a new printing machine cost in 5 years' time?
   c) What amount should the company budget for in 5 years' time?
   d) Calculate the monthly payment to be paid into the sinking fund, paying 12,5% p.a. compounded monthly, in order to have enough money to replace the printer in 5 years' time (the first payment is made immediately, and the last payment is made at the end of the 5 years).

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Answers:

1) 84

2).
a) R53 244,64
b) R184 634,87
c) R131 390,23
d) R1 552,43

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<tr><td>ACTIVITIES/ASSESSMENT</td><td colspan="2">Mind Action Series: Ex. 3- Pg 283; Ex. 4- Pg 285; Ex. 5- Pg 286; Ex. 6- Pg 293</td></tr>
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<tr><td></td><td colspan="2">Siyavula: Ex. 3- Pg 283; Ex. 6- Pg 293</td></tr>
<tr><td>CONSOLIDATION</td><td colspan="2">

- For a Loan use the Present Value Formula
- Note the Conditions for Present Value Formula
- Read the question carefully and apply correct formulae

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